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AMERICAN JOURNAL of PHYSICS

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A Graphical Demonstration of White Light Interference

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(Received January 23, 1952)

A teaching device is described which can be used in lectures on the interference of white light. The wave forms of four selected components of white light, in the violet, green, yellow, and red regions, are shown one above the other on a cardboard chart, inked in their respective colors. A sheet of clear plastic with an identical set of waves drawn on it is placed over the chart so that the two sets are in register. The combination represents a beam of white light as it arrives at the retina. A retardation of part of the beam is illustrated by displacing the plastic sheet laterally. Since the sine waves differ among themselves in wavelength, the various phase changes which result among the four components are shown quite clearly.

The principle has been embodied in slide rule form with the wave trains appearing at a large plastic window. Five smaller windows indicate, respectively, path difference in millimicrons, dominant wavelength and purity of the resulting light when C.I.E. Illuminant *C* is used, and relative energies and resultant hue for the four components shown.

INTRODUCTION

SELECTIVE interference among the components of a beam of polychromatic light may be achieved in a number of ways, including thin film interference, slit and grating phenomena, birefringence of polarized light, etc. In elementary lectures on light and color, it is difficult to present a satisfactory picture of these effects in terms of wave phenomena because of limitations of student background. A graphical means to this end has been evolved which appears quite economical of background requirements.

The point is made, in the lecture, that the various interference phenomena which result in colored light are but separate means to a common end, that of causing a part of the beam of white light to traverse a greater optical path than the rest of the beam before arriving at the screen or, ultimately, the retina. Phase changes among various components at the destination are then illustrated by the device to be described.

THE DEMONSTRATION

In its simplest form the display consists of a rectangular sheet of white cardboard of convenient viewing size and a sheet of transparent plastic of the same size. The wave forms of four selected components of white light are shown one above the other on the cardboard sheet, extending across the width thereof. The sine wave trains are representative of the violet, green, yellow, and red spectral regions, and they are inked in their respective colors. Their relative wavelengths conform with actual representative wavelengths in these regions. An

* Now at Research Laboratory, American Optical Company, Stamford, Connecticut.

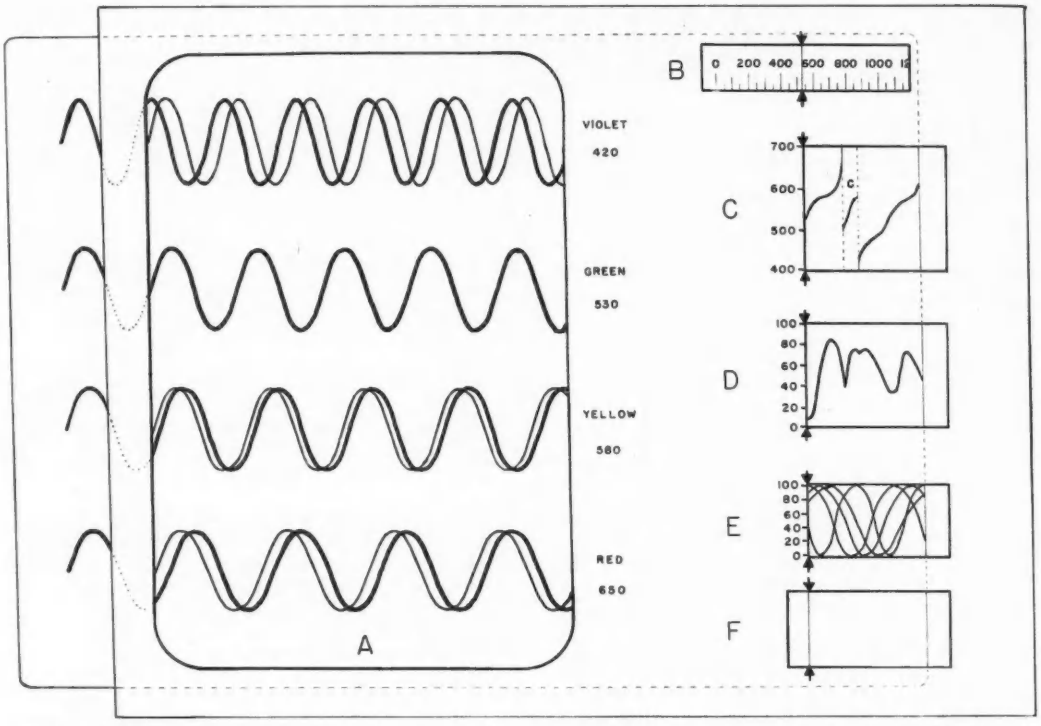

FIG. 1. Interference slide rule shown for optical path difference of 530 mμ corresponding to one complete "green" wave. Window *A* shows phase relationships among four selected components. Scales *B* through *F* are labeled as follows:

B—Optical Path Difference in Millimicrons
C—Dominant Wavelength for C.I.E. Illuminant *C*
D—Percent Purity for C.I.E. Illuminant *C*
E—Relative Energies for Four Components
F—Resultant Hue for Four Components.

Scale *F* is left blank in the figure. The scale indices are indicated by arrowheads.

identical set is shown on the plastic sheet such that it may be superimposed upon the first set.

When shown in registration, the two charts represent a beam of white light arriving coherently at the retina. A retardation of part of the beam is demonstrated by displacing the sheets laterally with respect to one another, whereupon the various phase changes become apparent. Since the four pairs of wave trains differ in wavelength from pair to pair, a retardation of one complete wave in the violet region, for example, will correspond to retardations of partial wavelengths in the other regions. This would amount to almost complete destructive interference of the "red" wave whose wavelength is nearly twice that of the violet.

REFINEMENTS

A more refined demonstration has evolved from this simpler form, offering the advantages of compactness and a more thorough treatment of the physical problem. The original cardboard chart is prepared as the slide of a slide rule with the plastic sheet mounted as a window in the body, as in Fig. 1. For various settings of the slide, the phase relationships are visible at the large window labeled *A* in the figure.

A sliding scale (*B*, Fig. 1) indicates the optical path difference corresponding to each position of the slide. The scale is drawn on the slide and is visible through a plastic window set into the body. The scale index is marked on the window and is denoted by arrowheads at the top and bottom.

In addition to the wave forms and the optical path difference scale, certain other information is provided which may be of interest in more detailed studies of interference color. This information is embodied in four additional scales. As with Scale *B*, these are drawn on the slide and indexed at their respective windows, the indices being aligned vertically with that of Scale *B*. On the actual slide rule, the five small scales indicated by letters in Fig. 1 are labeled as follows:

B—Optical Path Difference in Millimicrons
C—Dominant Wavelength for C.I.E. Illuminant *C*
D—Percent Purity for C.I.E. Illuminant *C*
E—Relative Energies for Four Components
F—Resultant Hue for Four Components.

Scales *E* and *F* refer to two hypothetical beams of light whose original spectral compositions, before interaction, include radiant energy only at the four wavelengths indicated in Scale *A*. Scales *C* and *D* refer to the interference effects in two equal beams of light whose spectral composition corresponds to C.I.E. Illuminant *C*. This standard light source has a continuous spectral energy distribution as contrasted with the line spectrum in the previous case and is similar in appearance to daylight.

THE *E* AND *F* SCALES

At any setting of the slide, inspection of Scale *A* will reveal to some extent the relative amplitudes which may be expected among the four components as a result of interference. For the position indicated in Fig. 1, for example, corresponding to an optical path difference of 530 mμ, the resulting amplitudes are seen to be at a maximum in the green and to fall off in the order of yellow, red, and violet due to the increasing phase differences. The progression of amplitudes is based upon visual estimates of the algebraic sum of each pair of wave trains. These estimates at best cannot yield a precise evaluation of relative amplitudes. Moreover, since it is energy, or amplitude squared, rather than amplitude which is perceived by the eye as a measure of brightness, the relationships among the four resultant amplitudes are of no direct psychophysical interest.

It is the purpose of Scale *E* to afford a more explicit interpretation of the various phase relationships. This scale is labeled "Relative Energies for Four Components" and shows four trigonometric curves representing energy as a function of optical path difference. Each curve corresponds to one of the component radiations of Scale *A* and, on the slide rule, is inked in its corresponding color for identification purposes. Assuming that the two interacting beams are identical and that equal energies are represented at the four wavelengths, it is clear that for zero path difference the relative energies of the four components after interaction will be at 100 percent of their maximum values. As path difference is increased from zero, the four resultant energies begin to fall off unequally until the first minimum, in the violet, is reached. Thereafter, the remaining curves in turn reach their minima and begin to rise. Those portions of the curves between zero path difference and 530 mμ are not visible in the figure.

The trigonometric curves of Scale *E* are of the form

$$I \propto \cos^2(\Delta/2),$$

which has the following significance. We assume two beams of monochromatic light which are identical in phase, energy, and wavelength λ. If there should be imposed upon the beams a geometrical path difference of d mμ in a medium whose index of refraction is n, the optical path difference between beams will be nd and the phase difference Δ will be given by $\Delta = 2\pi nd/\lambda$. When the beams are recombined the relative intensity I as a function of Δ is indicated by the expression above. The intensity will be a maximum when the wave trains coincide and vice versa.

For each of the four values of λ in Scale *A*, a curve has been plotted in Scale *E*. The wavelengths of the curves are adjusted to accord with the units of nd in Scale *B*.

An obvious application of Scale *E* is the preparation of a Scale *F* to show the resultant hue at each position of the slide. The hue at each value of optical path difference is derived from the relative energies at that position. In preparing Scale *F* it would be assumed, of course, that the original light contained equal amounts

of the four selected radiations only. Scale *F* is labeled "Resultant Hue for Four Components" and in the original model was prepared with colored pencil. No attempt was made to reproduce saturation or relative brightness. The scale is left blank in the reproduction.

THE *C* AND *D* SCALES

The usefulness of the slide rule is not limited to the case where the beam of light is composed of only four spectral lines. In fact, the value of the four-line spectrum is purely pedagogic and it was desired to generalize the treatment by including the case of the continuous spectrum. However, the interference colors in a beam of light of a continuous spectral nature will not necessarily be the same as those arising from light of a discontinuous nature even though both sources may appear white to the eye. Hence, neither Scale *E* nor Scale *F* will be applicable to the continuous spectrum. Furthermore, white light of continuous spectral composition does not have a unique meaning. There is a wide variety of spectral energy curves representing sources which will appear essentially white to the eye, and there is no *a priori* reason for these white sources to yield identical colors under identical conditions of interference.

It was therefore necessary to decide upon some specific source of white light of general laboratory interest, such as C.I.E. Illuminant *C*, and to have information describing the interference colors as a function of Δ for two equal beams of light originating in this source. The interference color information is available in the technical literature and will presently be dealt with in some detail.

It was also necessary to select some means of color specification which would be more specific than that of Scales *E* and *F*. From the point of view of the physicist, a numerical system of color specification is to be preferred, and for that purpose several excellent conventions are available. Of these, the designation "dominant wavelength and excitation purity" is perhaps most closely akin to the subjective sensory aspects of a given color stimulus. This mode of designation more than compensates in preciseness for what it lacks by not depicting the colors visually as does Scale *F*.

CHROMATICITY DIAGRAMS

Interference colors are seen "by their own light" as it were and fall into the category of self-luminous colors or colored light sources. They are to be distinguished from non-self-luminous or surface colors which are apt to be more subject to changes in appearance due to changes in background and surrounding illumination.

A complete physical description of a colored light requires a specification in terms of three independent variables such as dominant wavelength, purity, and luminance. By relating these variables to three independent coordinates in a Euclidean three-space, it is possible to represent lights of all colors in a so-called "color space" wherein each point designates a particular color. In certain cases of color measurement, dominant wavelength and purity are often of more particular interest than luminance or luminance differences. In such a case the luminance aspect of color may often be treated separately or even ignored entirely. This is generally true in the study of interference colors where dominant wavelength and purity are treated jointly under the heading "chromaticity" and where luminance considerations are of secondary importance.

The entire gamut of chromaticities is readily represented by a plane figure in the form of a graph, each point of which may be specified by only two variables, such as dominant wavelength and purity. Such a graph is known as a chromaticity diagram and may be thought of as a plane in the color space perpendicular to the luminance axis. Each point in the chromaticity plane may be thought of as representing an entire family of color sensations which differ among themselves only in luminance.

There is considerable latitude in the choice of coordinates when a chromaticity diagram is to be prepared. The respective coordinates need not represent dominant wavelength and purity explicitly, and there are indeed certain advantages in another treatment which has been widely adopted. This method makes use of the principle that a given sensation of chromaticity may be reproduced by the proper mixture of any three primary light sources which are mutually independent, i.e., no one of which may be

matched in color by a mixture of the other two. If it is agreed to ignore luminance differences between the given color and the mixture of primaries, then the absolute amounts of the matching primaries may be ignored in favor of two ratios among the three quantities, reducing the number of variables from three to two. These ratios are usually chosen as the respective amounts of two of the primaries relative to the total amount, expressed decimally. The ratio expressing the amount of the third primary per total amount is superfluous since the three ratios sum to unity and hence only two of them are independent.

It is thus possible to construct a graph in which one axis refers, for example, to the relative amount of green in the matching primary mixture and the other to the relative amount of red. Each color sensation would then be represented by a point in the plane, and the proximity of this point to the origin of axes would indicate the relative amount of blue primary in the matching mixture. While the location of each point is expressed most simply in terms of the coordinate axes, it is possible to find a coordinate transformation so that each point is uniquely identified with a particular dominant wavelength and purity.

The disposition of chromaticities throughout the diagram will depend upon the choice of primaries. The position of a certain color sensation on one chromaticity diagram will generally differ from its position on another diagram whose axes are based on a different choice of primaries. It is therefore possible to devise as many different chromaticity diagrams as there are choices of primary sources.

THE C.I.E. CHROMATICITY DIAGRAM

The chromaticity diagram adopted by the International Commission on Illumination[1,2] in 1931 has become the most prominent means of communicating chromaticity data between workers in color measurement. This diagram is the product of an elaborate system of color specification standardized by the C.I.E. The

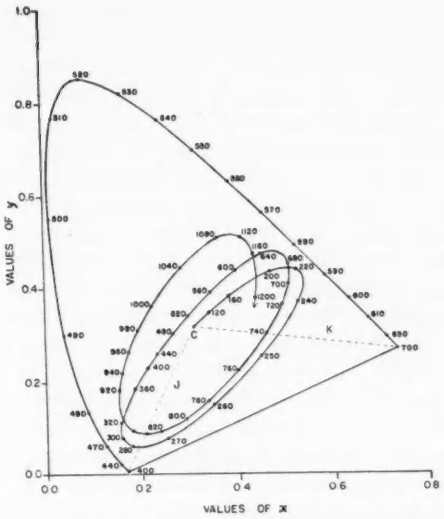

FIG. 2. The C.I.E. chromaticity diagram showing interference color reflected by a thin film on a glass surface as optical path difference increases from zero to 1200 mμ. All wavelengths are in mμ.

system is amply described in the literature[3–5] and will be discussed here only to the extent that it pertains to the derivation of Scales *C* and *D*.

The data from which these scales were derived are included in the C.I.E. chromaticity diagram of Fig. 2. The x and y axes denote, respectively, the amounts of red and green C.I.E. primaries, relative to the total, which are needed to match a particular color. The convoluted curve contains the interference color data used in preparing Scales *C* and *D*. This curve shows the chromaticities of interference colors as a function of optical path difference.

In addition there are depicted certain colors of especial interest. The point labeled *C* denotes the position of C.I.E. Illuminant *C* whose coordinates are $x=0.310$ and $y=0.316$. The horseshoe-shaped part of the enclosing curve is the locus of the spectral colors, the wavelengths of which are indicated at ten-mμ intervals. The straight line connecting the ends of the horseshoe is the boundary of the family of purple colors; wavelengths have no direct meaning here. This line

[1] Commission Internationale de l'Éclairage, Proceedings of the Eighth Session, Cambridge, England, 1931.
[2] Abbreviated "C.I.E." here and abroad, from the French translation.

[3] Donald J. Lovell, Am. J. Phys. **18**, 104 (1950).
[4] D. B. Judd, J. Opt. Soc. Am. **23**, 359 (1933).
[5] A. C. Hardy, *Handbook of Colorimetry* (The Technology Press, Cambridge, Massachusetts, 1936).

together with lines J and K denote a triangle which encloses the family of purple colors.

There is one feature of the C.I.E. chromaticity diagram which makes it of tremendous value in predicting the results of additive color mixtures. If we designate any two colors by their positions on the chromaticity diagram and connect these positions by a straight line, this line will be the locus of all possible additive mixtures of the two colors. The position of a given mixture along the line will depend upon the relative amounts of the two components. The locus of the purples is nothing more than a mixture line for the violet and red limits of the visible spectrum. The entire enclosure therefore represents the locus of the most highly saturated colors which have physical meaning. Within the enclosure lie all real colors and outside is the realm of supersaturated or imaginary colors.

DOMINANT WAVELENGTH AND EXCITATION PURITY

As an application of the additivity property of the C.I.E. chromaticity diagram, we note that it is possible to draw a straight line from the Illuminant C point through any given point in the enclosure until it intersects either the spectrum locus or the locus of purples. In terms of actual color mixing, this means that it is always possible to find a spectral color or purple which when diluted with Illuminant C will match a given chromaticity.

When a given color is matched by such a mixture, the wavelength of the spectral component of the mixture is defined as the dominant wavelength of the given color. When the chromatic component of the match is a purple rather than a spectral color, the dominant wavelength is specified in terms of the complement of the purple, or that spectral green whose mixture line with the purple passes through the Illuminant C point. The dominant wavelengths of the purple colors are indicated by adding the letter c after the numerical designation so as to distinguish them from their green complements.

It should at this point be apparent that as we move outward from point C along a straight line toward the spectrum locus, the colors represented become more saturated while remaining constant in dominant wavelength. The purity of a given color may be said to vary from zero at point C to 100 percent at the spectrum locus, assuming Illuminant C to be chosen as having zero saturation. More specifically, if we consider the distance from point C to the point representing a certain color, then the ratio of this distance to the total distance from point C to the spectrum locus is defined as the excitation purity of that color. Excitation purity is the psychophysical correlate of saturation and will be referred to here simply as purity.

On the C.I.E. diagram it is therefore possible to transform the chromaticity coordinates x and y of any color to a specification in terms of dominant wavelength and purity, the transformation involving a fairly simple graphical construction and numerical calculation. These latter steps are reduced to a matter of simple interpolation through the use of a series of specially constructed charts appearing in Hardy's *Handbook of Colorimetry*.[5]

INTERFERENCE COLORS

We have now formulated the basis for transforming the chromaticity data from x, y coordinates, in which they generally appear, to the more readily visualized dominant wavelength and purity. The remaining problem in the preparation of the slide rule is the one of selecting from the literature that set of interference color data which applies to the present case. Before this can be done, a distinction must be made between two general types of thin film interference, one being a special case of the other.

When a ray of light is reflected from the surface of a dense optical medium back into a less dense or rare one, a 180° phase change is introduced. This phenomenon is partly responsible for the phase difference in the beams of light reflected from the upper and lower surfaces of a thin film in air, the remainder of the phase difference being accounted for by any optical path difference present. As the film thickness becomes vanishingly small, the interfering beams become exactly out of phase, since one of the reflected beams has suffered a phase reversal while the

other has not. The following cases may be listed as examples of net phase changes due to reflection:

a. Reflection by a thin film in air.
b. Reflection by a thin air film between dense media.
c. Transmission by a thin film of intermediate refractive index bounded by one dense and one rare medium.

On the other hand, in many cases of interference either both of the reflected beams or neither of them happen to undergo phase reversals, in which case an infinitely thin film shows zero phase difference. This is illustrated by the following cases:

d. Transmission by a thin film in air.
e. Transmission by a thin air film between dense media.
f. Reflection by a thin film of intermediate refractive index bounded by one dense and one rare medium.

In the first group of examples, the sequence of interference colors as a function of optical path difference is distinctly different from the sequence resulting in the second group, the colors in the respective groups being complementary for a given optical path difference. The second group of phenomena, involving no net phase differences due to reflection, is of more general interest. It is under these conditions that the interference colors of thin films are very similar to those arising in birefringent and photoelastic phenomena and slit and grating phenomena. It is this group of phenomena rather than the first group which is treated in the present slide rule, for Scale *A* is so designed that for zero optical path difference the interfering beams are exactly in phase rather than out of phase.

THE INTERFERENCE COLOR DATA

Interference in thin films and birefringent media has been a topic of interest to many investigators since the time of Boyle. As a result of both early and contemporary investigations, the periodical literature abounds in data specifying interference color as a function of film thickness or optical path difference. A few of these investigations are listed chronologically.[6–10] The investigations are diverse in several respects and provide an adequate choice of data to fit various situations. Included in the data are the cases of phase changes arising from optical path difference alone as well as phase changes partly due to reflection by denser media. Most of the results are empirical, although recent papers have presented data derived on theoretical grounds. Interference colors are specified for various light sources as well.

From the wide choice of data available, the theoretically derived results of Kubota[10] appeared a sound choice on which to base Scales *C* and *D*. Kubota's data satisfy the requirement of phase coincidence for zero optical path difference since they are computed for the case of reflection by a thin film on glass. Illuminant *C* is the assumed light source, which makes the data especially useful in the general case.

As presented in Kubota's paper, the data are in the form of the convoluted curve shown in Fig. 2. The original parameter, optical film thickness, has been replaced here by optical path difference. The former is the product of the film thickness and its index of refraction, and, since the light was assumed normally incident upon the film, these values had only to be doubled to yield values of the new parameter. The refractive index of the film is assumed constant at all wavelengths, thereby neglecting dispersion and widening the generality of the results for the present purpose.

Readers familiar with the C.I.E. chromaticity diagram will recognize the progression of colors represented by the locus shown in Fig. 2. The continuity of the curve is of interest, as well as the repeated spectral orders. Starting at point *C*, we imagine an increasing thickness of film to be deposited on the glass surface as we trace out the locus of the interference colors. The light reflected from the surface takes on a pale yellow appearance at first, becoming more saturated as the locus approaches the spectral wavelength

[6] Lord Rayleigh, *Scientific Papers* (The University Press, Cambridge, England, 1886), Vol. II, p. 498.
[7] R. V. Baud and W. D. Wright, J. Opt. Soc. Am. **20**, 381 (1930).
[8] Eberhard Buchwald, Ann. Physik **38**, 5, 325–339 (1940).
[9] J. A. Prins and J. J. M. Reesinck, Physica **12**, 396 (1946).
[10] Hiroshi Kubota, J. Opt. Soc. Am. **40**, 146 (1950).

of 580 mμ. As the optical path difference increases from 200 mμ to 240 mμ the reflected light becomes straw colored and then copper colored, having reached a maximum of saturation just after passing through 220 mμ. A further increase in film thickness changes the color to reddish brown and then to purple at an optical path difference of about 260 mμ. A change through violet, pale blue, and a very pale green brings the locus close to point *C* again, nearly closing the first spectral order. A second order ensues and then part of a third, bringing the locus to the limit of Kubota's investigation. It is well known that a further increase in film thickness will yield a few more spectral orders of decreasing saturation until the locus returns ultimately to point *C* whereupon no further interference color is seen. It is for this reason that interest is confined to thin films when interference color is concerned.

By stereoptican means, the x and y coordinates of various points on the locus were found directly from Kubota's figure. These coordinates were then transformed to values of dominant wavelength and purity through Hardy's charts which are conveniently prepared for Illuminant *C*. The resulting triple column of figures, showing dominant wavelength and purity as a function of optical path difference, is represented graphically in Scales *C* and *D*, respectively. The slide rule designations for these scales are "Dominant Wavelength for C.I.E. Illuminant *C*" and "Percent Purity for C.I.E. Illuminant *C*."

In Scale *C* of Fig. 1, a discontinuous segment of the curve will be noticed. This segment, labeled "*c*," designates those interference colors whose dominant wavelengths are the complements of the spectral greens between 492 and 567 mμ. There are two such segments in Scale *C*, the first one not visible in the figure. Each segment corresponds to a segment of the Kubota locus which lies in the purple region of the chromaticity diagram. The discontinuities occur at the intersections of the locus and lines *J* and *K* and are responsible also for discontinuities in the slope of the purity curve of Scale *D*. It is recalled that as far as perception is concerned, the interference colors change in a perfectly continuous manner with changing path difference. The discontinuities in the curves are merely curious features arising from the manner in which dominant wavelength and purity are defined here.

DETAILS OF CONSTRUCTION

The present slide rule, constructed of sheets of stiff cardboard and clear plastic, measures 20×28 inches and is large enough to be seen throughout the average classroom. The optical path difference scale is so constructed that ½ inch corresponds to 100 mμ. The six-inch range of the slide thus accounts for path differences up to 1200 mμ, the limit of information available from Kubota's paper. Using the above scaling, the four wave trains were drawn with wavelengths corresponding to 420, 530, 580, and 650 mμ, as indicated in Fig. 1.

The stationary set of wave trains and the indices for the five scales were drawn on a sheet of 0.010-inch clear acetate which was then mounted on the body of the slide rule directly over the slide. A sheet of 0.060-inch clear Plexiglas mounted over the acetate serves to protect the figures on it and provides rigidity. A sheet of heavy white paper with appropriate windows cut in it serves as the face of the slide rule and provides the surface for the labeling of the various windows.

An adaptation which is here suggested is the preparation of the slide rule in the form of a pair of color slides for projection purposes. In this form the demonstration would be suitable for large audiences.

ACKNOWLEDGMENT

The development of the present slide rule from the simple wave demonstration was undertaken at the suggestion of Dr. Isay Balinkin whose interest and many suggestions are gratefully acknowledged.

Two Ballistics Problems of Future Transportation

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(Received April 14, 1952)

Two ballistics problems that may arise in future transportation have been formulated and solved in detail. The first problem is that of a vehicle moving along a great circle of the earth; the second problem is that of a rocket moving radially away from the earth. In both cases the magnitude of the resultant acceleration applied to the body has been set equal to a constant. The value of this acceleration will be determined ultimately by the maximum acceleration that human beings or material objects can withstand.

A PRIMARY function of transportation is to convey human beings rapidly from place to place with a minimum amount of discomfort. In general, for maximum transportation efficiency, the highest possible velocity would be maintained at all times. Nature, as formulated in nonrelativistic physics, prescribes no upper limit to possible velocity. On the other hand, nature imposes a rigid upper limit to change of velocity, or *acceleration*. Consider some common examples illustrating this effect of inertia:

1. The maximum acceleration of an elevator is determined by the amount of physical discomfort that the passengers will tolerate.

2. An airplane requires a certain minimum amount of time and space to accelerate to its take-off speed or to decelerate to a stop.

3. A jet plane or a rocket is limited in its minimum turning radius and maximum thrust.

4. Maximum muzzle velocity in ordnance is determined by the material strength of the projectile and of the weapon.

Assume that the following function and its first derivative are single-valued and continuous:

$$s=s(t), \quad \left.\begin{matrix} s=0 \\ \dot{s}=0 \end{matrix}\right\} \quad \text{when} \quad t=0, \tag{1}$$

where the dot signifies differentiation of distance s with respect to time t. Write the identities

$$\dot{s}\equiv\frac{ds}{dt}, \tag{2}$$

$$\ddot{s}\equiv\frac{d^2s}{dt^2}, \tag{3}$$

$$\ddot{s}\equiv\frac{d\dot{s}}{dt}, \tag{4}$$

$$\ddot{s}\equiv\frac{d\dot{s}}{ds}\frac{ds}{dt}, \tag{5}$$

$$\ddot{s}\equiv\frac{d\dot{s}}{ds}\dot{s}. \tag{6}$$

Separate variables in Eq. (6) and integrate:

$$\dot{s}^2\equiv\int_0^s 2\ddot{s}ds. \tag{7}$$

Substitute Eq. (2) in Eq. (7) and integrate:

$$t\equiv\int_0^s\frac{ds}{\left[\int_0^s 2\ddot{s}ds\right]^{\frac{1}{2}}}. \tag{8}$$

Equations (7) and (8) are well known in mechanics. Assume that $\ddot{s}$ in Eq. (8) is constrained to the range

$$0\leqq\ddot{s}\leqq\ddot{s}(t)\leqq c. \tag{9}$$

From the meaning of the definite integral, t is minimized when $\ddot{s}$ takes on its maximum value at all times. In the following problems, let the tangential acceleration $\ddot{s}$ be maximized subject to the condition that the resultant of all accelerations applied to the body shall not exceed a constant c in magnitude.

THEORY OF THE TRANSIT TUBE

In order to minimize transportation time between two points on a planet, subject to a maximum tolerable acceleration c, future man probably could not improve upon a *nearly straight*

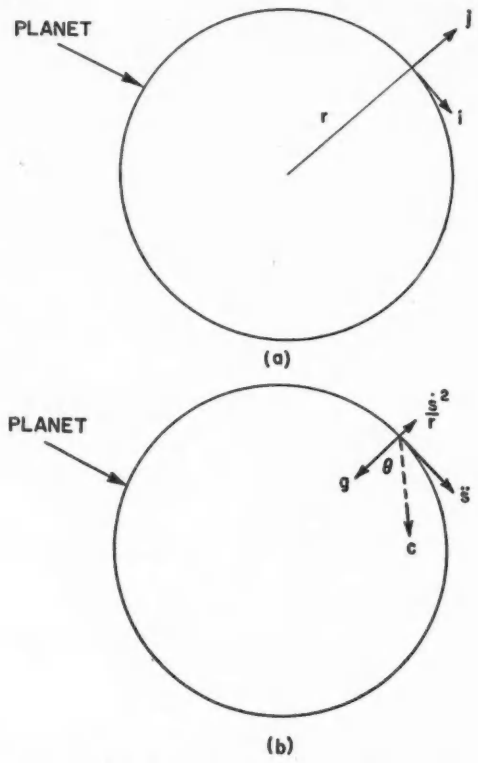

FIG. 1. (a) Coordinate system of transit tube. (b) Accelerations acting on transit tube vehicle moving with velocity $\dot{s}$.

tube in which his vehicle would travel. Future man will encounter difficulties if he should attempt to sink such a tube to great depths, for example, through the center of the earth. Circumstances probably will force a compromise, that is, the construction of a *curved* tube along the circumference of the earth. Consider a body that starts from rest and proceeds along a great circle of the earth with a tangential acceleration $\ddot{s}$ such that the resultant of all accelerations applied to the body has a constant value **c**. In addition to the tangential acceleration, the existing accelerations are those due to gravity and centrifugal force. The small, but not negligible, acceleration of Coriolis will not be considered. Consider a rectangular coordinate system, as shown in Fig. 1(a), attached to the vehicle, where **j** is normal to the earth. Formulated mathematically, see Fig. 1(b), the problem reduces to a solution of the following differential equation with the associated boundary conditions:

$$\ddot{s}\mathbf{i}+(g-\dot{s}^2/r)\mathbf{j}=\mathbf{c}, \quad \left.\begin{matrix} s=0 \\ \dot{s}=0 \end{matrix}\right\} \text{ when } t=0, \quad (10)$$

where g and r represent, respectively, the average acceleration due to gravity, and the average radius of the planet. Equation (10) can be written in scalar form

$$\ddot{s}^2+(g-\dot{s}^2/r)^2=c^2. \quad (11)$$

The dots in Eqs. (10) and (11) signify differentiation of distance s along the tube with respect to time t. For future reference, differentiate Eq. (11) with respect to time to obtain the relations

$$\frac{d\ddot{s}}{dt}=\frac{2\dot{s}}{r}\left(g-\frac{\dot{s}^2}{r}\right), \quad (12)$$

$$\frac{d^2\ddot{s}}{dt^2}=\frac{2\ddot{s}}{r}\left(g-\frac{3\dot{s}^2}{r}\right). \quad (13)$$

With respect to a perpendicular to the surface of the earth, the direction of **c**, Fig. 1(b), can be

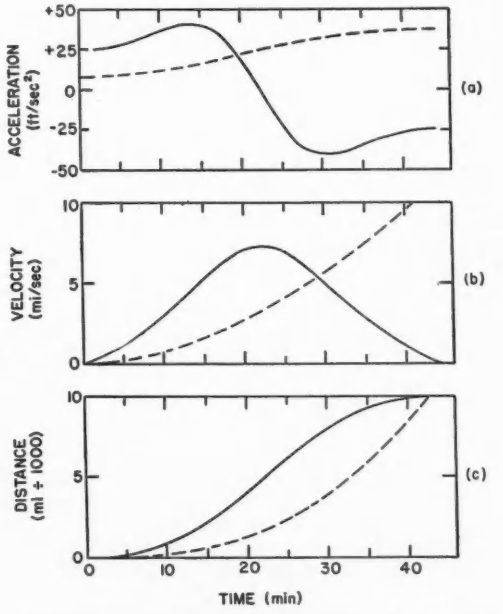

FIG. 2. Solution of the ballistics problems described in this paper. Solid lines represent the motion of the transit tube vehicle; dashed lines represent the motion of the vertical rocket.

represented by the relation

$$\theta \equiv \cos^{-1}\frac{\mathbf{c}\cdot\mathbf{j}}{-c} = \cos^{-1}\frac{g-\dot{s}^2/r}{c}. \tag{14}$$

Differentiate Eq. (14) with respect to time,

$$\dot{\theta} = 2\dot{s}/r. \tag{15}$$

Equation (15) states that the angular velocity of the resultant **c** is twice the angular velocity of the vehicle with respect to the center of the earth.

By the use of Eq. (6), a first integral of Eq. (11) is obtained:

$$\frac{ds}{d\dot{s}} = \frac{\dot{s}}{[c^2-(g-\dot{s}^2/r)^2]^{\frac{1}{2}}}, \tag{16}$$

$$s = -\frac{r}{2}\int_0^{\dot{s}} \frac{-2\dot{s}r^{-1}d\dot{s}}{[c^2-(g-\dot{s}^2/r)^2]^{\frac{1}{2}}}, \tag{17}$$

$$s = \frac{r}{2}\left(\sin^{-1}\frac{g}{c} - \sin^{-1}\frac{g-\dot{s}^2/r}{c}\right). \tag{18}$$

Write Eq. (11) in the form

$$(d\dot{s}/dt)^2 = c^2-(g-\dot{s}^2/r)^2. \tag{19}$$

Separate variables and integrate to obtain the relation

$$t = \int_0^{\dot{s}} \frac{d\dot{s}}{[(c-g+\dot{s}^2/r)(c+g-\dot{s}^2/r)]^{\frac{1}{2}}}. \tag{20}$$

Make the substitution

$$\dot{s} = [r(c+g)]^{\frac{1}{2}}\cos\phi \tag{21}$$

and transform Eq. (20) into the form

$$t = \left(\frac{r}{2c}\right)^{\frac{1}{2}}\int_{\phi_1}^{\pi/2} \frac{d\phi}{(1-k^2\sin^2\phi)^{\frac{1}{2}}}, \tag{22}$$

where

$$\phi_1 = \sin^{-1}\left[1-\frac{\dot{s}^2}{r(c+g)}\right]^{\frac{1}{2}}, \tag{23}$$

$$k^2 = \frac{c+g}{2c}. \tag{24}$$

Equation (22) is in the form of Legendre's standard elliptic integral of the first kind and can be evaluated readily by means of a table of elliptic functions,

$$t = (r/2c)^{\frac{1}{2}}[F(\pi/2, k) - F(\phi_1, k)]. \tag{25}$$

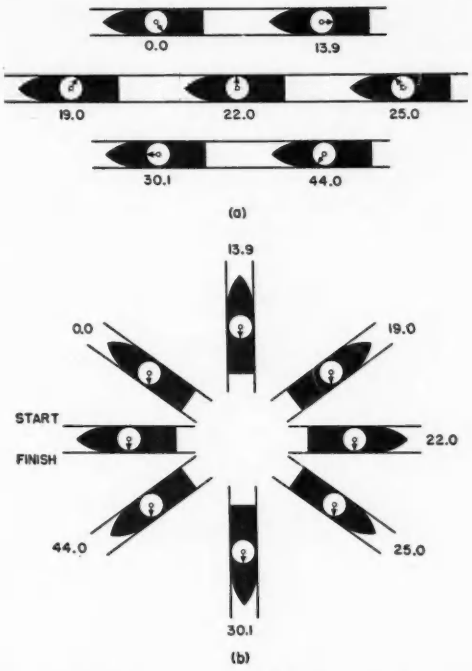

FIG. 3. Relative orientation of **c**. The numbers refer to time in minutes. 3(a). With respect to horizontal. 3(b). With respect to observer seated in a swinging chair free to orient with reference to **c**.

RESULTS

Equations (11), (18), and (25) represent the solution of the problem of minimizing transportation time along a great circle of a planet subject to a constant resultant acceleration **c**. Figure 2 represents a plot of these equations for a hypothetical trip determined by the following constants applicable to earth: $c=40$, $g=32.16$, $r=20\,891\,200$. (The units are English.) The solid lines of Fig. 2 represent the motion of a vehicle that accelerates to maximum velocity in the first 5000 miles and decelerates to a stop in the next 5000 miles. The curves are symmetrical about their mid-point. Such a hypothetical trip is equivalent to that from New York City, USA, to Darwin, Australia, via the Aleutian Islands. The direction of the constant resultant **c**, indicated by arrows in Fig. 3(a), is the direction that an observer in the vehicle would consider "down." Assume that an observer is seated in a swinging chair free to orient with reference to the constant

TABLE I. Values for transit tube at critical points in Fig. 2.

Point	$\sin\phi_1$ (Eq. 23)	$\ddot{s}$ (ft/sec²)	$\dot{s}^2/r$ (ft/sec²)	s^* (ft)	θ (Eq. 14)
1	$+1$	$+(c^2-g^2)^{\frac{1}{2}}$	0	$\frac{r}{2}(\Phi-\Phi)$	$+\cos^{-1}\frac{g}{c}$
2	$+\left(\frac{c+\frac{2}{3}g}{c+g}\right)^{\frac{1}{2}}$	$+[c^2-(\frac{2}{3}g)^2]^{\frac{1}{2}}$	$\frac{g}{3}$	$\frac{r}{2}\left(\Phi-\cos^{-1}\frac{-\frac{2}{3}g}{c}\right)$	$+\cos^{-1}\frac{\frac{2}{3}g}{c}$
3	$+\left(\frac{c}{c+g}\right)^{\frac{1}{2}}$	$+c$	g	$\frac{r}{2}\left(\Phi-\frac{\pi}{2}\right)$	$+\frac{\pi}{2}$
4	$+\left(\frac{c-g}{c+g}\right)^{\frac{1}{2}}$	$+(c^2-g^2)^{\frac{1}{2}}$	$2g$	$\frac{r}{2}\left(\Phi-\cos^{-1}\frac{g}{c}\right)$	$+\cos^{-1}\frac{-g}{c}$
5	0	0	$c+g$	$\frac{r}{2}(\Phi)$	π
6	$-\left(\frac{c-g}{c+g}\right)^{\frac{1}{2}}$	$-(c^2-g^2)^{\frac{1}{2}}$	$2g$	$\frac{r}{2}\left(\Phi+\cos^{-1}\frac{g}{c}\right)$	$-\cos^{-1}\frac{-g}{c}$
7	$-\left(\frac{c}{c+g}\right)^{\frac{1}{2}}$	$-c$	g	$\frac{r}{2}\left(\Phi+\frac{\pi}{2}\right)$	$-\frac{\pi}{2}$
8	$-\left(\frac{c+\frac{2}{3}g}{c+g}\right)^{\frac{1}{2}}$	$-[c^2-(\frac{2}{3}g)^2]^{\frac{1}{2}}$	$\frac{g}{3}$	$\frac{r}{2}\left(\Phi+\cos^{-1}\frac{-\frac{2}{3}g}{c}\right)$	$-\cos^{-1}\frac{\frac{2}{3}g}{c}$
9	-1	$-(c^2-g^2)^{\frac{1}{2}}$	0	$\frac{r}{2}(\Phi+\Phi)$	$-\cos^{-1}\frac{g}{c}$

* $\Phi=\cos^{-1}-g/c$.

resultant acceleration **c**. He interprets the changing direction of **c** as rotation of the vehicle about his stationary seat. Figure 3(b) represents the sketches of Fig. 3(a) oriented in accordance with the viewpoint of the observer. As in Fig. 3(a), the arrows indicate the direction of the constant resultant **c**. The observer experiences an increase in his weight of about 25 percent; an individual who normally weighs 161 lb would seem to weigh 200 lb for the duration of the trip.

Conditions at critical points that exist on the solid curves of Fig. 2, are illustrated in Fig. 3. General values of physical conditions that exist at nine such points are listed in Table I. Many of these quantities, pertinent to the hypothetical trip of Fig. 2, are listed in convenient numerical units in Table II. The nine critical points are characterized by the following conditions:

1. To avoid passenger distress caused by sudden impact of force, the initial acceleration $\ddot{s}$ is applied gradually with a rise time of a few seconds. The initial point $t=0$ represents conditions a moment after the entire acceleration $\ddot{s}$ has been switched on, and before the vehicle has moved appreciably. The resultant acceleration **c** is at an angle of about 36° with the perpendicular to the earth, Fig. 3(a). The observer interprets this condition as rotation of the vehicle through an angle of about 36°, Fig. 3(b). In terms of the hypothetical New York–Darwin trip, the vehicle begins to move out of the New York terminal toward the northwest with an initial tangential acceleration of about 24 ft/sec².

2. When $t=9.2$ min, the acceleration curve, Fig. 2(a), assumes an inflection, see Eq. (13). The centrifugal force at this point is $\frac{1}{3}$ the force of gravity. This point occurs as the vehicle passes over Lake Superior en route to Canada.

3. When $t=13.9$ min, the centrifugal force equals the force of gravity. Under these condi-

tions the vehicle would tend to circle the earth with neither a tendency to fall nor a tendency to recede to a greater distance from the earth. This point occurs as the vehicle passes a few hundred miles northeast of Edmonton, Canada.

4. When $t=19.0$ min, the vehicle has acquired escape velocity; that is, should the vehicle be released from its constraint, it would recede to infinity with respect to the earth. This point occurs as the vehicle passes over the Alaska Peninsula. The vehicle has passed the northernmost point of its trajectory and has begun to turn south.

5. When $t=22.0$ min, the vehicle, moving to the southwest a few hundred miles south of the Aleutian Islands, has acquired its maximum velocity. The tangential acceleration $\ddot{s}$ has been reduced to zero and the acceleration curve, Fig. 2(a), passes through an inflection point, see Eq. (13). Assume that subsequent to this point the vehicle begins to decelerate. This assumption is equivalent to the requirement that Eq. (12) be continuous. Equation (11) would still hold, however, if the acceleration $\ddot{s}$ were held at zero for some time subsequent to this point and prior to deceleration. At this point the resultant acceleration **c** is directed radially away from the earth, Fig. 3(a). The observer interprets this condition as rotation of the vehicle through an angle of 180°, Fig. 3(b).

6. When $t=25.0$ min, the distance s, see Table I, is independent of c. This point on the trajectory occurs approximately half-way between Midway Island and Tokyo, Japan.

7. When $t=30.1$ min, the vehicle is a few hundred miles south of Guam.

8. When $t=34.8$ min, the vehicle leaves North New Guinea toward Australia.

TABLE II. Numerical values for transit tube at critical points in Fig. 2.

Point	t (min)	$\ddot{s}$ (ft/sec²)	$\dot{s}$ (mi/sec)	s (mi)	θ (deg)
1	0.0	+23.7	0.00	0	36
2	9.2	+33.8	2.83	730	58
3	13.9	+40.0	4.89	1842	90
4	19.0	+23.7	6.95	3684	144
5	22.0	0.0	7.33	4940	180
6	25.0	−23.7	6.95	6216	216
7	30.1	−40.0	4.89	8058	270
8	34.8	−33.8	2.83	9170	302
9	44.0	−23.7	0.00	9900	324

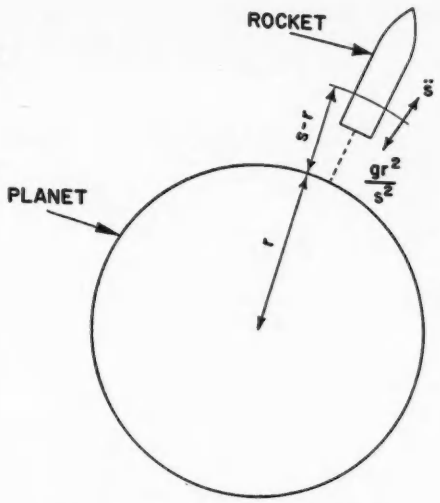

FIG. 4. Geometry of the radial rocket.

9. When $t=44.0$ min, the vehicle has ended its trip. The observer interprets conditions at this point as rotation of the vehicle through an angle of about 324°, Fig. 3(b). In practice, the tangential acceleration $\ddot{s}$ would be switched off with a delay time of a few seconds to avoid passenger distress caused by sudden release of weight.

THEORY OF THE VERTICAL ROCKET

A problem similar to that of the transit tube involves a rocket that starts from rest, rises radially from a planet, and recedes to infinity against the force of gravity which diminishes as the inverse square of the distance from the center of the planet. In order to minimize transportation time, set the resultant acceleration equal to a maximum constant value c. Formulated mathematically (see Fig. 4) the problem reduces to the solution of the following differential equation with the associated boundary conditions:

$$\ddot{s}+gr^2s^{-2}=c, \qquad \left.\begin{matrix} s=r \\ \dot{s}=0 \end{matrix}\right\} \quad \text{when} \quad t=0, \tag{26}$$

where g and r represent, respectively, the average acceleration due to gravity and the average radius of the planet. Integrate Eq. (26) and introduce the boundary conditions to obtain the relation

$$\tfrac{1}{2}\dot{s}^2=cs+gr^2s^{-1}-cr-gr. \tag{27}$$

Equation (27) can be written in the form

$$\frac{1}{2}\left(\frac{ds}{dt}\right)^2=\frac{cs^3-r(c+g)s^2+gr^2s}{s^2}. \tag{28}$$

Separate variables in Eq. (28) and rearrange to obtain the relation

$$dt=\left(\frac{2r}{c}\right)^{\frac{1}{2}}\frac{(s-0)ds}{(r-0)^{\frac{1}{2}}[4(s-r)(s-gc^{-1}r)(s-0)]^{\frac{1}{2}}}. \tag{29}$$

Equation (29) is in a form that can be integrated.[1] Substitute into Eq. (29) the relation

$$s=r\sin^{-2}\phi \tag{30}$$

to obtain the relation

$$dt=\left(\frac{2r}{c}\right)^{\frac{1}{2}}\frac{-d\phi}{\sin^2\phi(1-k^2\sin^2\phi)^{\frac{1}{2}}}, \tag{31}$$

where

$$k^2=g/c. \tag{32}$$

Equation (31) can be written in the form

$$dt=\left(\frac{2r}{c}\right)^{\frac{1}{2}}\left[(1-k^2\sin^2\phi)^{\frac{1}{2}}-(1-k^2\sin^2\phi)^{-\frac{1}{2}}-\frac{1-k^2\sin^4\phi}{\sin^2\phi(1-k^2\sin^2\phi)^{\frac{1}{2}}}\right]d\phi. \tag{33}$$

Integrate Eq. (33) to obtain the relation

$$t=\left(\frac{2r}{c}\right)^{\frac{1}{2}}\left[\int_{\pi/2}^{\phi_1}(1-k^2\sin^2\phi)^{\frac{1}{2}}d\phi-\int_{\pi/2}^{\phi_1}\frac{d\phi}{(1-k^2\sin^2\phi)^{\frac{1}{2}}}+\cot\phi_1(1-k^2\sin^2\phi_1)^{\frac{1}{2}}\right], \tag{34}$$

where

$$\phi_1=\sin^{-1}(r/s)^{\frac{1}{2}}. \tag{35}$$

Equation (34) can be evaluated by means of a table of elliptic functions.[2] Write Eq. (34) in the form

$$t=\left(\frac{2r}{c}\right)^{\frac{1}{2}}\left\{K-F(\phi_1,k)-E+E(\phi_1,k)-\left[\left(\frac{r}{s}\right)^{-1}-1\right]^{\frac{1}{2}}\left[1-\left(\frac{g}{c}\right)\left(\frac{r}{s}\right)\right]^{\frac{1}{2}}\right\}. \tag{36}$$

Equations (26), (27), and (36) represent the solution of the problem of minimizing transportation time for a rocket in vertical ascent subject to a constant maximum acceleration c. These equations are plotted in dashed lines, Fig. 2, for a hypothetical trip determined by the following constants applicable to earth: $c=40$, $g=32.16$, $r=20\ 891\ 200$. As in the previous problem, an observer with an initial weight of 161 lb would appear to weigh 200 lb for all time during which the rocket was accelerating. Only two points are considered critical, that is, the inflection point of the acceleration curve, Fig. 2(a), and the point where escape velocity has been attained.

The point of escape can be determined from the relation

$$\dot{s}^2s^{-1}=2gr^2s^{-2}. \tag{37}$$

Eliminate $\dot{s}$ between Eqs. (27) and (37) to obtain the relation

$$s=(1+gc^{-1})r. \tag{38}$$

Substitute Eq. (38) into Eq. (37) to obtain the escape velocity

$$\dot{s}=\left(\frac{2grc}{c+g}\right)^{\frac{1}{2}}. \tag{39}$$

Escape velocity is attained at the point $t=27.3$ min. Of secondary interest is the distance s at which the curve of Fig. 2(a) has an inflection point,

$$s=\frac{3r}{5}\left\{1+\frac{g}{c}+\left[1-\frac{17}{9}\left(\frac{g}{c}\right)+\left(\frac{g}{c}\right)^2\right]^{\frac{1}{2}}\right\}. \tag{40}$$

This inflection point occurs at the point $t=17.9$ min.

CONCLUSIONS

The two problems solved in this paper represent special cases of a single generalized problem,

[1] Edwin P. Adams and R. L. Hippisley, *Smithsonian Mathematical Formulae and Tables of Elliptic Functions* (The Smithsonian Institution, Washington, D. C., 1939), first edition, p. 256.

[2] G. W. Spenceley and R. M. Spenceley, *Smithsonian Elliptic Functions Tables* (The Smithsonian Institution, Washington, D. C., 1947), first edition.

that is, the problem of a vehicle that moves along an arbitrary path away from a planet subject to a constant resultant acceleration c. In problems of this kind, the imposition of constant resultant acceleration seems to be a realistic engineering requirement. Consequently, the results presented herein might well reflect a solution of future transportation problems. The final engineering problem must contain Coriolis corrections and allow for rate of increase of acceleration.[3]

[3] The importance of time derivatives of acceleration in mechanical and transportation problems has been considered by P. LeCorbeiller, Am. J. Phys. **13**, 156 (1945).

The Definitions of Magnetic Flux Density and Field Intensity

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The present theory of magnetism, in which it is treated as a phenomenon of current electricity, suffers from objections which are pointed out. The meanings of **B** and **H** on an atomic scale are considered, and a method of picturing **B** and **H**, inside a ferromagnetic body, without mentioning magnetic poles or dipoles, is developed. It is shown that this method satisfactorily accounts for the demagnetizing field in certain simple cases and that it also makes clear why **H** occurs in the circuital theorem while **B** is used in Laplace's equation. Rigorous definitions of **B** and **H** at any point in a magnetic medium are then developed. In the concluding section the advantages of the suggested method are summarized.

I. INTRODUCTION

At one time, all the definitions of magnetic quantities were based on the concept of an isolated pole. These definitions, including those of magnetic induction or flux density **B** and magnetic field intensity **H** inside a solid ferromagnetic material, were clear-cut and precise.[1] The system, however, suffered from certain objections. More recent theory, which treats magnetism as a branch of current electricity, overcomes these objections, but as usually presented it suffers from the following defects: (a) The definitions of **B** and **H** at a point in a solid ferromagnetic body are given in a very imprecise form. (b) Largely as a result of (a), it is often difficult to tell why **B** occurs in some results and **H** in others. (c) In arrangements involving ferromagnetic materials, the distribution of **H** is often difficult to reconcile with a distribution due solely to currents.

The aim of this paper is to give a theory free from these objections. The present definitions of **B** and **H** are considered, to illustrate the objections. By considering the atomic model, a method of overcoming these objections is found, and this leads to rigorous definitions of **B** and **H** at any point, without mention of magnetic poles.

Units and Abbreviations

Rationalized MKSC units are used. Symbol μ_0 = permeability of vacuum. The abbreviation *CC* is used to denote "circulating current."

II. THE ATOMIC CIRCULATING CURRENT AND THE EQUIVALENT DIPOLE

Magnetism is frequently considered as due to atomic *CC*s. This idea of an atomic *CC* needs clarification. Each atom is assumed to contain many electrons, revolving in orbits and spinning on their axes, which must give rise to magnetic effects similar to those due to currents. For most purposes the total magnetic effect of an atom can be considered the same as that of a current flowing in a closed loop. It is this equivalent current in a closed loop which is meant by the atomic *CC*.

The magnetic effects of a *CC* are, at a distant point, the same as those of a magnetic dipole. This process of replacing a *CC* by an equivalent dipole is a useful mathematical device, but no physical significance should be attached to the

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[1] G. Burniston Brown, Proc. Phys. Soc. (London) **52**, 577 (1940).

magnetic poles thus introduced. It should, moreover, be possible to define **B** and **H** without the necessity of introducing poles by this device.

III. PRESENT DEFINITIONS OF FLUX DENSITY

1. Flux Density B in Air

There are three possible methods of defining **B** at a point in air, *viz.*, (B1) in terms of the force experienced by a current element; (B2) in terms of the couple experienced by a small current loop; and (B3) in terms of the charge circulated (or emf induced) when the flux linkage with a small coil is altered by rotating the coil, or by withdrawing it to a point of known **B**, or by altering **B** to some known value.

Method (B1) is the commonest definition. Textbooks rarely use (B3) as a definition, since considerable work is usually done on magnetic fields before induced emf's are considered. But it would require only minor rearrangement to reverse this order and use (B3) as the definition of **B**. The outstanding advantage of this is that (B3) is the usual method of measuring **B** in practice, not only in air but also in ferromagnetic bodies, whereas the other two methods are of little use when ferromagnetic bodies are considered.

2. Flux Density B Inside a Solid Ferromagnetic Body

Modern theory is very weak on this definition. If the body is in the form of a bar, or an anchor ring, a coil wound on the body can be used to give an *average* value of **B** over the cross section, by (B3) above. This is adequate for most practical purposes, and is as far as most textbooks go. The determination of **B** *at any specified point* in the body is usually not considered. Smythe[2] suggests that a spherical cavity can be excavated at the point, filled with a liquid of the same permeability as the solid, and that a measuring apparatus (e.g., a rotating coil) placed in the liquid. This could be used as a theoretical definition, but details of a method of determining when the match of permeabilities was achieved would be needed.

[2] W. R. Smythe, *Static and Dynamic Electricity* (McGraw-Hill Book Company, Inc., New York, 1939).

IV. PRESENT DEFINITIONS OF FIELD INTENSITY

1. Magnetic Field Intensity H in Air

For **H** at a point in air, there are two very differently worded, but almost identical, definitions, *viz.*: (H1) The field intensity is equal to the flux density at the point divided by μ_0:

$$\mathbf{H}=\mathbf{B}/\mu_0.$$

(H2) The field intensity is given by the integration of a specified function over all the currents causing the field

$$\mathbf{H}=(1/4\pi)\oint i[d\mathbf{s}\times\mathbf{r}]/r^3.$$

The unit "amp-turns/m" for **H** arises from this definition. This is the total **H** if magnetic material is absent, but if it is present this calculation gives only that part of **H** due to the external currents. Since the portion of **B** due to external currents is given by μ_0 times the same integral, this definition is only a variation of (H1).

2. Magnetic Field Intensity H inside a Solid Ferromagnetic Body

In a ferromagnetic body, the portion of **H** due to the external currents is calculated by (H2) above. In addition, there is a portion of **H** due to the magnetism of the body, which is usually in the opposite direction to **B** inside the body, and is called the demagnetizing field. The currently accepted theory gives no method of calculating this demagnetizing field for the general case. Method (a) below expresses the demagnetizing field in terms of another unknown variable, while method (b) deals with a particular case. These methods are given to point out their deficiencies.

(a) To every elemental volume of the body is assigned a magnetic moment, and the demagnetizing field is obtained by evaluating the field due to such a magnetic moment and integrating over the body. The connection of this process of assigning a magnetic moment to an elemental volume with the electric current theory of magnetism is based on the possibility of replacing the atomic *CC*s by magnetic dipoles. The magnetic moment per unit volume is the intensity of magnetization **M**, a vector function of position.

(a′) A variation of this method is to assign **M** as above and then obtain **H** at the point (total **H**,

not the demagnetizing field) from the equation[3]

$$\mathbf{B}=\mu_0\mathbf{H}+\mathbf{M}. \quad (1)$$

(b) This method considers the special case of a cylindrical bar of ferromagnetic material, of finite length, uniformly magnetized in a direction parallel to the axis of the bar.[4] All the atoms are assumed to have *CC*s of the same magnitude, and their axes parallel. For one section through the bar, these atomic *CC*s are shown in light lines in Fig. 1, and the method suggests that these can be replaced by a single *CC* round the outside of the bar, shown in heavy line in Fig. 1. The entire bar can thus be replaced by an equivalent solenoid. The number of amp-turns/m of this equivalent solenoid is chosen to give the same **H** at one point outside the bar as is actually given by the magnetized bar. Then **H** at other points outside the bar, and also the torque on the bar when placed across a weak magnetic field, are correctly given by this equivalent solenoid. It is readily shown that by attributing equal and opposite poles to the ends of the bar, along with plausible assumptions about the action of poles, one can equally well determine both the external **H** and the torque. The demagnetizing field at any point inside the bar is now calculated *from these poles*, completely ignoring the equivalent solenoid.

It will be seen that methods (a) and (a′) cannot give the demagnetizing field unless the value of **M** at every point of the body can be found. Method (b) only deals with a special case, so that none of the methods can find the demagnetizing field in the general case. But even accepting this fact, there are other objections to these definitions: (1) Both (a) and (b) find it necessary to introduce poles. In (a) the word "pole" may not be mentioned, but magnetic dipoles are introduced either explicitly or by obtaining the mathematics by analogy with dielectric theory, which is based directly on electric dipoles. In (b) the poles are only an alternative for the external field, but become an essential part of the calculation of the demagnetizing field. Method (a′) might be worded to avoid poles, but it reduces Eq. (1) to the status of a definition,

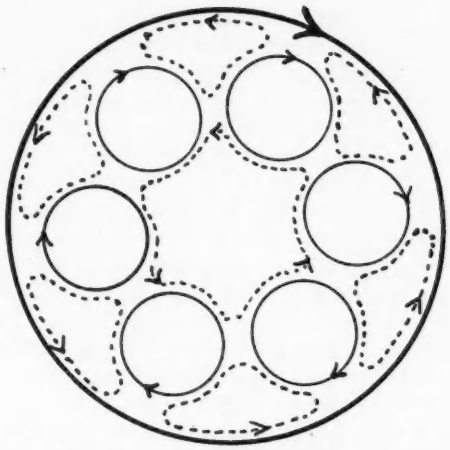

FIG. 1. Circulating currents in a magnetized bar. Light lines: atomic circulating currents; heavy line: principal circulating current; dotted lines: reverse circulating currents.

thereby removing any possibility of deducing this equation. (2) If the bar of method (b) is permanently magnetized, then **H** is in the same direction as **B** outside the bar, but in the opposite direction inside the bar, the change of direction of **H** occurring suddenly at the end faces of the bar. Such a sudden change of direction does not occur for **H** of the equivalent solenoid, and the method fails to explain this discrepancy. This objection is equally applicable to (a), since the methods have much in common. For this bar, method (a) would give **M** constant, and the Poisson distribution equivalent to the set of dipoles is then a pole on each end of the bar, equal and opposite, agreeing with method (b). The difference between the methods is that (a) immediately replaces the atomic *CC*s by poles (in the form of dipoles) and then sums the effects, while (b) attempts to sum the atomic *CC*s and then converts the result to a distribution of poles.

It appears, therefore, that both methods pay lip service to the idea of *CC*s, but ultimately have to resort to the idea of poles; and both produce a result which they cannot explain in terms of currents only.

V. A SUGGESTED ATOMIC INTERPRETATION OF B AND H: BASIC PRINCIPLES

The above definitions show that **B** has physical significance (affecting forces on currents, induced

[3] Or $\mathbf{B}=\mu_0(\mathbf{H}+\mathbf{M})$, depending on the definition of **M**.

[4] It is recognized that an elongated spheroid is more amenable to mathematical treatment, but for descriptive work the uniformly magnetized cylinder is more convenient.

emf's) but they do not explain why **H** is needed. The need for **H** arises when ferromagnetic materials are studied. When an anchor ring of the material is wound with a toroidal primary and a small secondary coil, a change in primary current produces a change in **B**, the average of which over the cross section is measured by the secondary coil. Flux density **B** thus obtained varies in a nonlinear manner with the primary current, and an extension of the experiment shows that it is the amp-turns/m of the primary on which **B** depends. This quantity, the amp-turns/m of the primary, is given a name and symbol, **H**. Since for a toroid wound in air **B** is always proportional to **H**, as just defined, it is reasonable to assume that for cases in air where **B** varies from point to point, **H** also varies and is a field vector.

It is usual to represent **B** by lines of flux drawn so that their density (lines/m^2 or webers/m^2) gives the magnitude of **B**. It is also usual to draw lines of **H** with a similar convention, but in the following a different convention is used. *Lines of H will be drawn so that their density* (lines/m^2) *is* μ_0 *times the magnitude of H.* This has the advantage that, in air, lines of flux can be considered as either lines of **B** or lines of **H**. In a ferromagnetic body the convention will be maintained that the lines of **H** have a density μ_0 times the magnitude of **H**, but lines of **B** and of **H** will no longer coincide.

In this and the next section we must imagine that we can see the atomic *CC*s, like wire loops carrying currents, and see the lines of flux like pieces of string threading these *CC*s. After a clear picture has been obtained, by this process, of what represents **B** and what **H**, we can go on to consider how **B** and **H** can be measured when we cannot "see" lines of flux but have to rely on instrument readings.

On this picture, the ferromagnetic body is replaced by a set of atomic *CC*s in vacuum. If **B** and **H** could be observed on this scale, they would both vary rapidly from point to point, with lines of flux representing both **B** and **H** and with $\mathbf{H}=\mathbf{B}/\mu_0$. But when we speak of **B** and **H** in a ferromagnetic body, we refer to averages over an area large compared to atomic dimensions, so that the local variations around the atoms do not appear in the results. If this averaging were carried out in a straightforward manner **H** would still remain equal to $\mathbf{B}/\mu_0$. The fact that this relationship is not observed for a ferromagnetic body shows that the method of averaging to find **H** is not the obvious one, and we must try to find what method is used. To do this, we return to the case of the toroid on an anchor ring, for which we defined **H** as the amp-turns/m of the primary.

For this special case, **H** is the same whether the anchor ring is made of ferromagnetic material or of air. Since the effect of a ferromagnetic body is due to the atomic *CC*s, the above fact suggests that in obtaining **H** from the lines of flux, we should count only selected lines of flux, which pass where the effect of the atomic *CC*s is least, which is in the gaps between atoms. The inference is that in a block of ferromagnetic material of arbitrary shape, the value of **H** at any point is obtained by counting the number of lines of flux passing, in a certain small area, through gaps between the atoms in that area, and dividing by the area and by μ_0.

The next simplest case is that of the short magnetized bar of method (b), Sec. IV, which we shall at first consider to be permanently magnetized (no external currents). Assuming for simplicity that successive cross sections all have their atoms in line with those of Fig. 1, each such line of atoms will be similar to a line of coaxial coils. Some lines of flux will thread through the *CC* of each atom in turn along the line, giving continuous lines of flux in the general direction of **B**. These will be lines of **B**. But each atomic *CC* will also have some lines of flux which form closed loops round the current. These closed loops have the same general direction as **B** inside the atomic *CC*s, but the opposite direction outside them, between the lines of atoms. As lines of **H** are observed between the atoms, they must be identified with these portions of closed loops between the lines of atoms. Thus, our assumption, i.e., that in obtaining **H** only lines of flux passing between atoms are to be counted, leads immediately to an explanation of the fact that **H** inside the material is in the opposite direction to **B**.

It must be realized that at this point in the discussion the ideas are only qualitative. As there is no means of knowing how many closed loops of flux should be associated with each atomic *CC*, no numerical value can be given for **H** even when

B and the geometry are fixed. The quantitative aspects of the problem are discussed in the next section. For the moment we shall assume that the total number of lines of flux due only to the atomic *CC*s is fixed and invariant, whether a current is flowing in an external solenoid or not. This is a reasonable assumption as long as the external current is not sufficient to cause realignment of the atomic *CC*s.

If now a current in an external solenoid is applied, and gradually increased in the direction to assist **B**, more and more of the lines of flux will flow continuously through successive atoms, reducing the number of closed loops of flux and hence the reverse field intensity, until all the closed loops are absorbed ($\mathbf{H}=0$).[5] Further increase of the current will force some continuous flux lines between the lines of atoms, giving **H** in the same direction as **B**. Figure 2 gives sketches of these three conditions, illustrating the way in which **B** and **H** are viewed on the ideas advanced above.

It is interesting to compare the behavior of lines of **B** and of lines of **H** for such a bar when permanently magnetized. A line of **B**, followed from an arbitrary point in air, will proceed until it enters the bar; then, according to the above description, it will continue through the bar, *threading its way through successive atoms*, until it emerges from the bar and continues through air to its starting point.[6] Starting from the same point in air, the line of **H** follows the same flux line as the line of **B**, until it reaches the bar. The line of flux continues into the bar, but it is no longer called a line of **H**; the line of **H** stops short on the surface of the bar. Inside the bar are some closed lines of flux linking atomic *CC*s, and the portions of these between lines of atoms, where they are in the opposite direction to **B**, are counted as lines of **H**. In other words, lines of **H** outside the bar stop on the face of the bar and have no connection with the lines of **H** inside the bar, which in turn are not continuous but are only selected portions of complete, closed lines of flux.

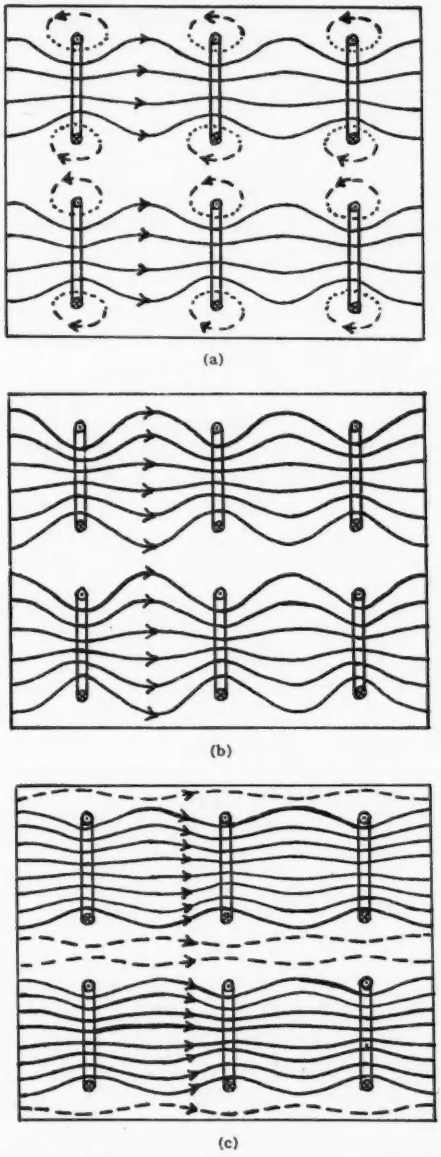

FIG. 2. Axial cross section through bar at various stages of magnetization. (a) No current in magnetizing solenoid; (b) solenoid current adjusted to give $\mathbf{H}=0$ in bar; and (c) solenoid current greater than in (b). Solenoid current flows clockwise when viewed in direction of **B**.

Six atomic currents are shown in cross section; a dot and a cross represent current flowing out of and into the paper, respectively. Full lines, with arrowheads, are lines of flux which are counted as lines of **B**; dotted lines are parts of lines of flux which are not counted for either **H** or **B**; and dashed lines are parts of lines of flux which are counted as lines of **H**.

[5] This statement relies on the assumption that the total number of lines of flux due to an atomic *CC* is fixed and independent of the current in the external solenoid.

[6] Or to near its starting point, as J. Slepian (Am. J. Phys. 19, 87 (1951)) has shown. The difference does not affect our argument in any way.

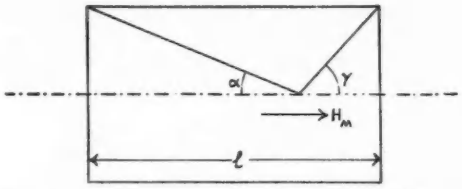

FIG. 3. Axial cross section through cylindrical bar magnet, showing symbols used in Sec. VI, Part 2. The principal circulating current, of (Ni) amp-turns, is flowing like a current in a solenoid round the bar, counterclockwise when viewed in the direction of $\mathbf{H}_M$.

This picture explains the sudden change of direction of **H** at the face of the bar. In so doing, it makes it clear that **H** has little or no physical significance, but is introduced only because of the way we choose to specify the magnetic properties of ferromagnetic materials.

VI. A SUGGESTED ATOMIC INTERPRETATION OF B AND H: MATHEMATICAL RESULTS

Since the portions of **B** and **H** due to external fields are readily calculated, in the following only $\mathbf{B}_M$ and $\mathbf{H}_M$, the portions of **B** and **H** due to magnetism of the material, are considered. Earlier $\mathbf{H}_M$ was called the demagnetizing field. As already mentioned, when we speak of **B** and **H** in a solid body we refer to averages (taken in a special manner as explained above) over an area which contains thousands of atoms, but which is infinitesimal compared to the dimensions of the body. The next step is to progress from the atomic model to a picture with no smaller unit than these infinitesimal areas. During this step a portion of the body with the cross section shown in Fig. 1 will be considered, and we shall continue to assume that we can see the lines of flux, and hence can count their number.

For $\mathbf{B}_M$, counting the lines of flux crossing an area is the same as counting the lines threading continuously through lines of atoms, since the closed loops of flux can add nothing to the count. In method (b) of Sec. IV the atomic *CC*s (light lines of Fig. 1) were replaced by a single *CC* round the outside of the bar (heavy line of Fig. 1), hereafter called the **principal *CC***. This is equivalent to allowing the orbits of the atomic *CC*s to expand and distort until they touch at all points to form a complete mesh, while the value of the current decreases to maintain the number of continuous lines of flux constant. The number of amp-turns in the principal *CC*, which is chosen to give agreement with the experimentally observed **B**, is equal to the value for these expanded and distorted *CC*s.

But the replacement of the atomic *CC*s by the principal *CC* gives a picture which excludes $\mathbf{H}_M$. There should also be included **reverse *CC*s**, shown dotted in Fig. 1, in which the current is exactly equal and opposite to that in the principal *CC*. In obtaining $\mathbf{H}_M$ by counting lines within an infinitesimal area, only lines passing through these reverse *CC*s must be counted. If now the orbits of the atomic *CC*s are allowed to expand and distort as explained above, with consequent decrease of the value of the principal *CC* and the reverse *CC*s, then the reverse *CC*s within the infinitesimal area must also expand (and possibly distort) to keep the number of lines through them constant. *As a working hypothesis* we shall assume that they meet to give a complete mesh, so that they can be replaced by a single reverse *CC* round the infinitesimal area. Consequently, $\mathbf{H}_M$ is given by the number of lines passing through an infinitesimal area, divided by the area and by μ_0, when the *principal CC flows round the outside of the bar and an equal reverse CC flows round the infinitesimal area.*

Since for both principal *CC* and reverse *CC*, adjacent turns are separated by the interatomic distance while their cross sections have dimensions vastly greater than this, both principal *CC* and reverse *CC* can be considered as flowing in tightly wound solenoids. The amp-turns for both these currents will be denoted by (Ni). With this picture we can now consider some mathematical results.

1. Anchor Ring Sample, as in B-H Measurement

Both the principal *CC* and reverse *CC* flow in toroids, so that each produces a field intensity of $(Ni)/l$, where l is the mean length of the toroid. As these intensities are in opposite directions, $\mathbf{H}_M=0$.

The starting point of the present discussion was the fact that **H** in such a sample was just that due to external currents, so that $\mathbf{H}_M=0$. The fact that our modified picture of *CC*s has

produced this result is a justification of the working hypothesis given above.

2. Short, Uniformly Magnetized Bar

An axial cross section, explaining the symbols, is given in Fig. 3. The principal *CC* is a solenoid of finite length, giving a field intensity $\mathbf{H}_P$, where

$$\mathbf{H}_P=(Ni/2l)(\cos\alpha+\cos\gamma).$$

The reverse *CC* flows in a solenoid of infinitesimal cross section, and so gives a field intensity $\mathbf{H}_R=(Ni)/l$. Hence,

$$\mathbf{H}_M=\mathbf{H}_R-\mathbf{H}_P=(Ni/2l)\{(1-\cos\alpha)+(1-\cos\gamma)\}.$$

This result is precisely that which would be obtained by assuming a uniform density of pole strength on the two faces, so that it agrees with the result given by the older method.

3. The Circuital Relation

The circuital relation, *in air*, can be stated in two forms:

$$\oint\mathbf{H}\cdot d\mathbf{s}=\sum i, \quad \text{(a)} \qquad \oint\mathbf{B}\cdot d\mathbf{s}=\mu_0\sum i, \quad \text{(b)}$$

where $\sum i$ is the sum of all currents linked by the path of integration. If magnetic media are present anywhere on the path of integration, both results must still hold, but $\sum i$ must include the *CC*s linked by the path. For (a), the path passes inside both the principal *CC* and the reverse *CC*, which are equal and opposite, so the magnetic material adds nothing to $\oint\mathbf{H}\cdot d\mathbf{s}$. But for (b), the path links only the principal *CC*, so the magnetic material makes a big contribution to $\oint\mathbf{B}\cdot d\mathbf{s}$. Thus, (a) is correct even when $\sum i$ is redefined to be only the sum of currents flowing in actual circuits linked by the path of integration (i.e., omitting *CC*s), but (b) is not true under these circumstances if magnetic media are present. As $\sum i$ is usually given this definition, (a) is the correct form for the circuital relation.

4. Laplace's Equation (div$\mathbf{B}=0$)

The discussion in Sec. V makes it clear that lines of **B** are always continuous, while lines of **H** may be discontinuous at the surface of and within a ferromagnetic medium. Consequently, it is clear that div$\mathbf{B}=0$, but that div**H** may not be zero.

5. Boundary Conditions

Once Laplace's equation and the circuital relation have been established, the boundary conditions at an interface between media follow by standard methods.

6. Intensity of Magnetization

It is possible to give a definition of **M** entirely in terms of *CC*s and to deduce Eq. (1). It is also possible to modify the idea of principal *CC* (replacing it by a mesh of *CC*s, each unit in the mesh being an infinitesimal area but containing thousands of atoms) to cover the case when **M** is not constant. However, it seems convenient at this point to introduce poles *as a convenient mathematical device*, without physical significance, and to continue the theory in terms of poles.

So far we have assumed we could "see" and count lines of flux. This is convenient for deducing theoretical results, but in practice we must rely on instruments for determining **B** and **H**. Methods of defining **B** and **H**, at a point in a ferromagnetic body, precisely and without any assumptions about visibility of lines, are considered in the next section.

VII. MACROSCOPIC DEFINITIONS OF B AND H

All three methods of finding **B** in air require movement of the test apparatus. As this is impossible in a solid medium, the only alternative is to cut a cavity in the medium. The idea of test cavities has fallen into disrepute, but it is surely better to accept definitions involving cavities than to try to manage without definitions. It must be emphasized that these cavities are actual holes cut in the material, not theoretical "gaps between atoms" such as have been considered earlier. As the cavities contain air, the methods of Secs. III and IV for determining **B** and **H** in air can be used, and the remaining problem is to ensure that **B** or **H** in the cavity is the same as it was in the material.

The first choice for the cavity is a cylinder with its axis parallel to **B** or **H**. Removal of the material of the cavity will cause a reverse *CC* to flow on the cylindrical wall.[7] To measure **B**, the effect

[7] An alternative view is to say the material now has two boundaries, the outside and the cylindrical wall of the cavity, and the principal *CC* flows on both. It is of little importance which view is taken, as long as it is realized

of this reverse *CC* must be made as small as possible, which is achieved by making the length of the cavity infinitesimal, so that the number of amp-turns of the reverse *CC* is infinitesimal. To measure **H**, the reverse *CC* must be allowed to have its full effect, which is achieved by making the diameter of the cavity infinitesimal, so that the reverse *CC* flows in an effectively infinite solenoid.

Since a cavity of (relatively) large diameter can be used for **B**, there seems to be little objection to extending it sideways till it reaches the edge of the medium. It thus becomes a crevasse, and measuring apparatus can (at least in theory) be inserted and manipulated. A coil placed in the crevasse is in the wrong orientation to experience a couple, so that method (B2) cannot be used, but methods (B1) and (B3) are feasible, and of these (B3) is preferred for reasons already given. Hence, the definition of **B** inside a solid medium could be worded as follows:

The flux density at a point in a magnetic medium is a constant times the charge circulated when a small coil is moved from a crevasse cut at the point, the walls of the crevasse being normal to the flux density and an infinitesimal distance apart, to a point of zero flux density. The constant is the total resistance of the coil circuit divided by the effective area of the coil.

For measuring **H**, method (H1) is used. The cylindrical cavity of infinitesimal radius allows no room for manipulating test apparatus. But a solenoid whose cross section is a rectangle with one side infinitesimal also gives a field intensity at the center of $(Ni)/l$, so the cylindrical cavity can be replaced by one of rectangular section, and this can then be extended along its long dimension till it reaches the surface of the medium and becomes a crevasse. Owing to the orientation, the only suitable method of measuring **B** in this crevasse is (B2). The definition of **H** inside a solid medium could thus be worded:

The field intensity at a point in a magnetic medium is a constant times the torque on a small current-carrying coil placed in a crevasse at the point, the walls and floor of the crevasse being parallel to the field intensity and the walls being an infinitesimal distance apart. The constant is the reciprocal of the product of μ_0, the current in the coil, and the effective area of the coil.

In both the above definitions, the coil is assumed to have finite diameter but infinitesimal thickness, so that it can fit into the crevasse in only one way. If the conditions have to be obeyed rigorously the definitions can only be theoretical, but in some cases a crevasse wide enough to accept a flat spiral coil of fine wire could be tolerated. The main criterion would be whether or not a crevasse of this width introduced a reluctance appreciable compared to that of the rest of the magnetic circuit.

VIII. CONCLUSION

It is not suggested that the methods outlined in Secs. V–VII will replace the concept of magnetic poles, or dipoles, for purposes of calculation. The advantages claimed for the methods outlined are the following: (a) A much clearer mental picture of the meanings of **B** and **H** is obtained. The way in which a demagnetizing field can be produced by atomic *CC*s is shown, and in so doing the artificial nature of the concept of field intensity is stressed. (b) The reasons why **H** is used in the circuital theorem; why **B** is used in Laplace's equation; and why the boundary conditions use **B** for normal components and **H** for tangential components, are made clear. (c) Rigorous definitions of **B** and **H** at a point inside a solid magnetic medium are provided. (d) Poles are not mentioned in obtaining any of these results. They can be introduced when needed, but it is clear that they are only a convenient mathematical device, not physical entities.

A further advantage of the methods outlined is that the early introduction of the atomic picture, with the idea of lines of flux threading successively through lines of atoms, and with local distortions between atoms, lays a sound foundation for explanations of hysteresis and the idea of domains.

that the current on the cylindrical wall is exactly equal and opposite to that on the outside, so that for practical purposes it fulfils the same role as the reverse *CC* we have considered earlier.

A Classroom Antenna Experiment

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(Received May 14, 1952)

This article describes equipment for studying antenna patterns without the use of cumbersome outdoor apparatus. Patterns for numerous arrays, in good agreement with theory, are readily obtained. Construction of arrays observed by students is encouraged.

THIS article describes an apparatus designed to conduct antenna pattern experiments in college laboratory space. The equipment is not cumbersome nor does it use frequencies which might interfere with present TV channels.

To place the receiver in the "far zone" use is made of a 12-cm wavelength. Lighthouse tube oscillators operating at this wavelength are frequently used in microwave optics experiments and may be purchased from the Central Scientific Company. Such equipment makes desirable the use of antennas as transmitting units. By reciprocity (except possibly for some of the "V" antennas) a receiving antenna pattern is the same as that of an identical transmitting antenna.

The transmitting unit shown in Fig. 1 is easily constructed at small expense. The turntable, 18 inches in diameter, is cut from plywood and mounted on the square plywood table by means of a bearing (New Departure No. 3203) and drawer rollers. Over-all height of the unit is about 24 inches. The scale is cut from bristol board, marked at one-degree intervals, and is glued to the turntable. To measure the angle of rotation a movable marker (not shown) is used. A two-prong microphone connector, centered on the axis of rotation, provides a termination into which the antenna can be slipped. A section of 300-ohm transmission line an odd number of quarter wavelengths long is used to couple the antenna to the oscillator.

The coupling loop is a piece of spring wire about 2½ inches long bent into a "U." The two wires of the feed line are soldered to the ends of the loop. Proper insertion of the loop into the oscillator cavity is necessary to obtain maximum output but no effort is made to match the line to the antenna.

The receiver is an intensity meter consisting essentially of a receiving dipole, reflector plate, crystal rectifier, and a microammeter with a scale range of 0–20 microamperes. This receiver is available from the Central Scientific Company as a part of their microwave optics equipment.

To make sure that radiation reaching the receiver is due only to the antenna, a framed copper wire screen fastened to the base and extended to a sufficient height effectively shields the receiving meter from transmission line and oscillator cavity radiation.

A Yagi antenna mounted in the transmitter is shown in Fig. 1. (The darker background around the antenna is merely for contrast.) A pattern obtained by students from this antenna with the receiver mounted on a wooden tripod about 35

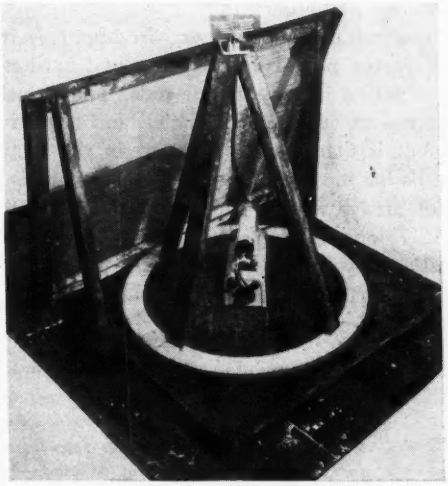

FIG. 1. Photograph of the transmitter.

* Now at the United States Air Force Institute of Technology, Wright-Patterson Air Force Base, Dayton, Ohio.

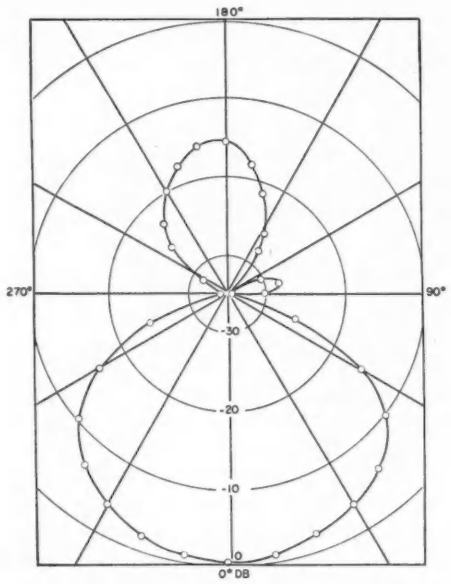

FIG. 2. Radiation pattern of a Yagi array. The radial coordinate is in *db*, that is $20 \log I/I_0$, where I_0 is the maximum current reading of the receiving microammeter, I the current reading at arbitrary azimuth. The asymmetry may be due to reflection from a wall.

wavelengths distant is shown in Fig. 2. A minor lobe in the 270° position is too small to show in the pattern on the scale selected. Reflections from objects and walls can affect the meter readings, and some care is necessary in the placement of apparatus and observers.

Antennas made of copper wire about $\frac{1}{16}$ inch in diameter, pulled to straighten and harden it, are mounted in polystyrene rods. The parasitic elements are inserted into holes, and the driven element is fitted into a slot cut in the mount. Details for the Yagi mount are shown in Fig. 3.

For arrays with a single vertical driven element, 75-ohm coaxial cable is used. The antenna is essentially an extension of the central conductor. Vertical parasites are placed in the desired position in a thin sheet of polystyrene. This sheet slips over the transmitting antenna and has a pattern of holes suitable for a variety of arrays.[1]

Often the most effective length of the parasitic elements is best determined by trial and error. Particularly is this true for the single parasitic director or reflector associated with a half-wave transmitting antenna. A tight fitting sleeve on the vertical antenna provides a means for changing its length. This is a convenience in determining the resonance length for the antenna.

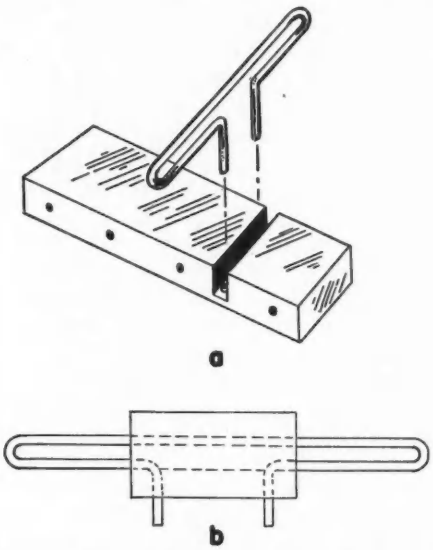

FIG. 3. (a) Details of the Yagi mount with folded dipole as fed element. (b) End view. The vertical segments are spaced to slip into the microphone connector. This figure is not drawn to scale.

The author is grateful to George Guthrie, a sophomore physics major at Wabash College, for constructing the transmitting unit and many of the antenna models.

[1] Bailey, *TV and other Receiving Antennas* (John F. Rider Publisher, Inc., New York, 1950). This book gives geometrical details of a great variety of arrays and is an excellent book to use in conjunction with this experiment.

Iron rusts from disuse, stagnant water loses its purity and in cold weather becomes frozen; even so does inaction sap the vigors of the mind.—LEONARDO DA VINCI.

Application of Dirac's δ-Function to Some Problems in Classical Physics

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The Dirac δ-function is used in solving the problem of bending of a beam and also some problems of vibration. It is seen that fairly general results can be obtained without complicated calculations.

THE function $\delta(x)$, introduced by Dirac,[1] has been extensively used in quantum mechanics. On account of its queer properties it has, so far, found little application in classical physics. The purpose of the present paper is to show that the δ-function can be used with advantage, in solving specially those problems of classical mechanics in which forces are concentrated at a point. As examples we shall consider (I) the problem of bending of a beam and (II) some problems of vibration. It will be noticed that fairly general results can be deduced in a very simple way with the use of the δ-function.[2]

As is well known, the δ-function is defined by the following two relations:

$$\int_{-\infty}^{\infty}\delta(x)dx=1, \tag{1a}$$

$$\delta(x)=0 \quad \text{for} \quad x\neq 0, \tag{1b}$$

i.e., $\delta(x)$ is zero everywhere except at the origin ($x=0$), where it tends to infinity in such a way that Eq. (1a) is satisfied. From this definition one can deduce a number of interesting properties of $\delta(x)$, but we state here only the two which we will use later in the calculations. They are

$$\int_{-\infty}^{\infty}f(x)\delta(x-a)dx=f(a), \tag{2}$$

$$\begin{aligned}\delta(x-a)&=0 \quad \text{for} \quad x\neq a,\\ &\to\infty \quad \text{for} \quad x=a.\end{aligned} \tag{3}$$

I. We first consider the bending of a beam of length l hinged at both ends and acted upon by a force P at a point $x=a$, with $(l>a>0)$. (We take one end of the beam as the origin.) Then the load density can be expressed as

$$W(x)=P\delta(x-a). \tag{4}$$

If Y is Young's modulus of the material of the beam and I is the moment of inertia of the cross section, then the displacement of the beam from its unperturbed position is given by the differential equation

$$YI\frac{d^4y}{dx^4}=-W(x)=-P\delta(x-a). \tag{5}$$

To solve this equation we make a Fourier analysis of $\delta(x-a)$. We write

$$\delta(x-a)=\sum_{n=1}^{\infty}a_n\sin\frac{n\pi x}{l}, \tag{6a}$$

where

$$a_n=\frac{2}{l}\int_0^l\delta(x-a)\sin\frac{n\pi x}{l}dx=\frac{2}{l}\sin\frac{n\pi a}{l}. \tag{6b}$$

Substituting for $\delta(x-a)$ from Eqs. (6) in Eq. (5) and integrating four times successively, we get—for the present boundary conditions—the solution

$$y=\frac{2Pl^3}{\pi^4YI}\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi a}{l}\sin\frac{n\pi x}{l}. \tag{7}$$

It will be noticed that the point a (where the force is applied) and the point x (where the displacement is observed) can be interchanged without changing the magnitude of the observed displacement.

On account of the factor $1/n^4$, the above series converges very rapidly; and even if we retain the first term only, the error involved is not large. We therefore take

$$y=\frac{2Pl^3}{\pi^4YI}\sin\frac{\pi a}{l}\sin\frac{\pi x}{l}. \tag{8}$$

* On Government of India Scholarship.
[1] P. A. M. Dirac, *Quantum Mechanics* (Oxford University Press, London, 1947), p. 58.
[2] Jointly with Dr. B. K. Agarwala, a preliminary note has already been published in *Physics Quarterly* **2**, 18 (1951).

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For all values of a ($l>a>0$) the maximum displacement of the beam occurs at the center, $x=l/2$. Taking $a=l/2$, we get

$$y_{\max}=\frac{2Pl^3}{\pi^4 YI}.$$

Apart from a small numerical factor, this agrees with the result usually deduced. (The small discrepancy is removed if instead of working with Eq. (8) we work with Eq. (7) and note that $\sum(1/n^4)=\pi^4/90$.) With the help of Eq. (7) one can also study the bending of a uniformly loaded beam. We replace P by Wda in Eq. (8) and integrate with respect to a from 0 to l, getting

$$y=\frac{4Wl^4}{\pi^5 YI}\sum_{n=1,3,5\cdots}\frac{1}{n^5}\sin\frac{n\pi x}{l},$$

where W is the load per unit length of the beam.

II. As a second example we consider the vibrations of a struck string. The differential equation of a vibrating string is

$$\frac{\partial^2 y}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}, \tag{9}$$

where $c^2=T/\rho$; ρ is the mass per unit length of the string; and T is the tension applied to it. The general solution of the above equation, for the case of a string of length l fixed at both ends, is

$$y=\sum_{n=1}^{\infty}\sin\frac{n\pi x}{l}\left(a_n\sin\frac{n\pi ct}{l}+b_n\cos\frac{n\pi ct}{l}\right). \tag{10}$$

If the string is excited by striking it at a point $x=a$, and if the blow imparts an "average" velocity v_0 to the string, we take as the initial conditions

$$y=0 \quad \text{for all } x \text{ at } \quad t=0,$$
$$\frac{\partial y}{\partial t}\equiv\dot{y}=lv_0\delta(x-a) \quad \text{at} \quad t=0.$$

These give

$$a_n=\frac{2l}{n\pi c}\int_0^l v_0\delta(x-a)\sin\frac{n\pi x}{l}dx$$
$$=\frac{2v_0 l}{n\pi c}\sin\frac{n\pi a}{l}, \tag{11a}$$
$$b_n=0. \tag{11b}$$

Substituting these in Eq. (10) we have, for the displacement,

$$y=\frac{2v_0 l}{\pi c}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi a}{l}\sin\frac{n\pi x}{l}\sin\frac{n\pi ct}{l}, \tag{12a}$$

and

$$\dot{y}=2v_0\sum_{n=1}^{\infty}\sin\frac{n\pi a}{l}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \tag{12b}$$

gives the instantaneous velocity at any point x of the string.

Forced oscillations of a string, when the force is concentrated at a point, can also be studied by making use of the δ-function.[3]

As a further example of the application of δ-function we consider the vibrations of a cantilever of length l, excited by striking it at any point $x=a$. The vibrations of a cantilever are represented by the differential equation

$$\frac{\partial^4 y}{\partial x^4}=-\frac{\rho}{YI}\frac{\partial^2 y}{\partial t^2}, \tag{13}$$

where ρ is mass per unit length of the beam. The general solution of this equation is[4]

$$y=\sum_{n=1}^{\infty}\psi_n(x)\left[B_n\cos\frac{2\pi nct}{l}+C_n\sin\frac{2\pi nct}{l}\right], \tag{14a}$$

where

$$\psi_n(x)=a_n\left[\cosh\frac{\pi\beta_n x}{l}-\cos\frac{\pi\beta_n x}{l}\right]$$
$$+b_n\left[\sinh\frac{\pi\beta_n x}{l}-\sin\frac{\pi\beta_n x}{l}\right], \tag{14b}$$

$$B_n=\frac{2}{l}\int_0^l y_0(x)\psi_n(x)dx,$$
$$C_n=\frac{1}{\pi\nu_n l}\int_0^l(\dot{y})_0\psi_n(x)dx, \tag{14c}$$

and a_n, b_n, and β_n are certain constants, while ν_n gives the allowed frequencies of vibration of the cantilever. If the cantilever is struck at a point $x=a$, so that the "average" velocity imparted to

[3] P. M. Morse, *Vibration and Sound* (McGraw-Hill Book Company, Inc., New York, 1936), p. 99.
[4] See reference 3, Chapter IV.

it is v_0, we have

$$B_n=0,$$

$$C_n=\frac{v_0}{\pi\nu_n}\psi_n(a).$$

The case when the cantilever is struck at the end is the one usually considered, and it is simple to show that here

$$\psi_n(a=l)=(-1)^{n-1}\sqrt{2}.$$

The above result is, however, more general, in that we can study the vibration of the bar excited by striking it at any point. If, for example, the cantilever is struck in the middle, $a=l/2$, then apart from the first no other odd overtones ($n=3, 5, 7, \cdots$) are excited.

As a final example we consider the vibration of a rectangular membrane of dimensions $a\times b$. A vibrating membrane is represented by the differential equation

$$\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}=\frac{1}{c^2}\frac{\partial^2\eta}{\partial t^2}, \tag{15}$$

where $\eta(x, y, t)$ is the displacement of the point (x, y) at the time t from its original unperturbed position. The solution of this equation is of the form

$$\eta=\sum_{m,n=1}^{\infty}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\{B_{mn}\cos 2\pi\nu_{mn}t + C_{mn}\sin 2\pi\nu_{mn}t\}, \tag{16a}$$

where

$$B_{mn}=\frac{4}{ab}\int\int\eta_0(x, y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy, \tag{16b}$$

$$C_{mn}=\frac{2}{\pi\nu_{mn}ab}\int\int v_0(x, y)\times\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy, \tag{16c}$$

and

$$\nu_{mn}=\frac{1}{2}\left(\frac{T}{\rho}\right)^{\frac{1}{2}}\left\{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right\}^{\frac{1}{2}} \quad \begin{matrix} m=1, 2, 3\cdots \\ n=1, 2, 3\cdots \end{matrix}. \tag{16d}$$

ρ is the mass per unit area of the membrane, and ν_{mn} represents the allowed frequencies of vibration. If the membrane is excited by striking it at a point (x_0, y_0), and if the impact imparts an average velocity v_0 to it then

$$B_{mn}=0,$$

$$C_{mn}=\frac{2v_0}{\pi\nu_{mn}}\sin\frac{m\pi x_0}{a}\sin\frac{n\pi y_0}{b},$$

and hence,

$$\eta=\frac{2v_0}{\pi}\sum_{m,n=1}^{\infty}\frac{1}{\nu_{mn}}\sin\frac{m\pi x_0}{a}\sin\frac{n\pi y_0}{b}\sin\frac{m\pi x}{a}\times\sin\frac{n\pi y}{b}\sin 2\pi\nu_{mn}t. \tag{17}$$

Any particular results can easily be deduced from the general expression (17).

The above few examples illustrate the use of the δ-function in solving certain problems of classical mechanics. Its use can be extended to more of such problems, where the forces are concentrated at a point. Further, as is clear from the problem of bending of a beam, even problems with uniform distribution of force can be treated, indirectly, by this method.

For example, tests show that voltages as high as 5000 volts can be developed by such common-everyday-occurrences as getting up from a plastic-covered chair. Oscillographic records, however, indicate that if the resistance between the person and the chair is less than 20 megohms, the voltages will not exceed 300 volts. This is less than the minimum sparking voltage in air and is thus not hazardous.—Hazards from Static Electricity, U. S. Bureau of Standards Technical Report.

The Eckart Conditions for a Polyatomic Molecule

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The Eckart conditions for a polyatomic molecule are discussed. They are introduced to insure the smallness of the interaction between vibration and rotation and define a system of coordinates in which the vibration of the atoms around their equilibrium positions is described.

FROM a completely general point of view a polyatomic molecule is a mechanical system formed by N nuclei and m electrons. To a high order of approximation, the motions of the electrons are independent of the motions of the atomic nuclei.[1] The nuclei, then, move in the electronic potential field and form a mechanical system with $3N$ degrees of freedom. This system is considered to be isolated, conservative, and mechanically stable, i.e., the interactions with other molecules are neglected, the forces derive from a potential, and there is an equilibrium configuration. There are a number of energy levels and corresponding states of this mechanical system which will be found by the solution of the corresponding Schrödinger equation and the transition between these states gives rise to a spectrum. The theory of molecular spectra deals with these two problems: the calculation of the energy levels or the frequencies, and the calculation of the transition probabilities. We will be interested only in the first problem here. The first step in this problem is finding the classical Hamiltonian of the polyatomic molecule to obtain later the quantum-mechanical Hamiltonian. The potential energy of the molecule depends only on the relative positions of the particles and is therefore invariant under a rotation or translation of the molecule as a whole. In the earlier work, the total energy of a molecule was considered to be the sum of an energy of vibration and an energy of rotation; but refined mathematical methods as well as empirical data indicated that this is not the case, and interactions between vibration and rotation had to be taken into account. In the mathematical treatment, the so-called Eckart conditions[2] are applied in such a way that this interaction will be small so that a perturbation treatment is possible in the quantum-mechanical formulation of the problem. The aim of this paper is to give a detailed derivation of the Eckart conditions following the approach of Sayvetz[3] and also to arrive at some of the formulas derived by Nielsen, Shaffer, and others[4] by a straightforward method and in greater detail than is given in the original papers.

If an atom is displaced in such a way that its relative position with respect to the other atoms is changed, restoring forces are called into play and thus the displacement from the equilibrium configuration will be small. Thus, the molecule behaves like a semirigid body which can translate and rotate freely while the nuclei move only slightly with respect to each other. In some cases, however, the relative motion of a part of the molecule with respect to other parts of the molecule may be considerable, as in the internal torsion of ethane or the inversion of nitrogen in ammonia; therefore, in general, we will have semirigid parts moving with respect to each other with their components moving only slightly from the equilibrium configuration and the whole structure still free to rotate and translate in space.

If the slight motion from the equilibrium configuration would be frozen, the parts of one molecule would be perfectly rigid, the potential energy would be identically zero, and the system would be left with only M degrees of freedom of which 6 (or 5 if the molecule is linear) describe the over-all rotation and translation in space, and the other $M-6$ (or $M-5$) describe the gross internal motions (like the torsion of ethane or the inversion of ammonia) when they are present.

[1] M. Born and J. R. Oppenheimer, Ann. Physik **84**, 457 (1927).
[2] C. Eckart, Phys. Rev. **47**, 552 (1935).
[3] A. Sayvetz, J. Chem. Phys. **7**, 383 (1939).
[4] H. H. Nielsen, Revs. Modern Phys. **23**, 90 (1951) and references given therein.

Then the Cartesian coordinates in space-fixed coordinate axes are functions of M coordinates $\alpha_p (p=1, 2, \cdots M)$, where α_p for $p=1, 2, \cdots$ describe the translation and rotation of the molecule as a whole and α_p for $p=7, \cdots M$ describe the gross internal motions. The equations of transformation will be

$$x_k = F_k(\alpha_p), \tag{1}$$

where $k=1, 2, \cdots 3N$.

The kinetic energy will be simply given by

$$2T = \sum_k m_k \dot{x}_k^2 = \sum_{kpp'} m_k \frac{\partial F_k}{\partial \alpha_p} \frac{\partial F_k}{\partial \alpha_{p'}} \dot{\alpha}_p \dot{\alpha}_{p'}$$

$$= \sum_{pp'} A_{pp'}{}^{o} \dot{\alpha}_p \dot{\alpha}_{p'},$$

where

$$A_{pp'}{}^{o} = \sum_k m_k \frac{\partial F_k}{\partial \alpha_p} \frac{\partial F_k}{\partial \alpha_{p'}};$$

or, in matrix form

$$2T = \dot{\alpha}_t A^o \dot{\alpha} = P_t A^{o\,-1} P,$$

where α is a column matrix formed by the α_p, α_t is its transpose, P is a column matrix formed by the momenta P_p conjugate to α_p, and $A^{o\,-1}$ is the reciprocal of A^o. Since the potential energy is identically zero, we can write the Hamiltonian as

$$2H = 2T = P_t A^{o\,-1} P.$$

If the atoms are now allowed to move slightly from their equilibrium configuration, the $3N-M$ degrees of freedom will reappear; let us call $q_j (j=1, 2, \cdots 3N-M)$ the generalized coordinates corresponding to this internal motion. Since the configuration departs only slightly from the rigid configuration described above, the Cartesian coordinates are considered to be linear functions of the q_j which are to be added to Eq. (1), and so

$$x_k = F_k(\alpha_p) + \sum_j f_{kj}(\alpha_p) q_j. \tag{2}$$

Let us now compute the Hamiltonian. The velocities are

$$\dot{x}_k = \sum_p \frac{\partial F_k}{\partial \alpha_p} \dot{\alpha}_p + \sum_{pj} \frac{\partial f_{kj}}{\partial \alpha_p} q_j \dot{\alpha}_p + \sum_j f_{kj} \dot{q}_j,$$

and the expression for the kinetic energy becomes

$$2T = \sum_k m_k \dot{x}_k^2 = \sum_{pp'k} m_k \frac{\partial F_k}{\partial \alpha_p} \frac{\partial F_k}{\partial \alpha_{p'}} \dot{\alpha}_p \dot{\alpha}_{p'}$$

$$+ \sum_{pp'jj'k} m_k \frac{\partial f_{kj}}{\partial \alpha_p} \frac{\partial f_{kj'}}{\partial \alpha_{p'}} q_j q_{j'} \dot{\alpha}_p \dot{\alpha}_{p'}$$

$$+ \sum_{pp'jk} m_k \left(\frac{\partial F_k}{\partial \alpha_p} \frac{\partial f_{kj}}{\partial \alpha_{p'}} + \frac{\partial F_k}{\partial \alpha_{p'}} \frac{\partial f_{kj}}{\partial \alpha_p} \right) q_j \dot{\alpha}_p \dot{\alpha}_{p'}$$

$$+ \sum_{jj'k} m_k f_{kj} f_{kj'} \dot{q}_j \dot{q}_{j'} + 2 \sum_{pjk} m_k \frac{\partial F_k}{\partial \alpha_p} f_{kj} \dot{\alpha}_p \dot{q}_j$$

$$+ 2 \sum_{pjj'k} m_k \frac{\partial f_{kj'}}{\partial \alpha_p} f_{kj} q_{j'} \dot{\alpha}_p \dot{q}_j,$$

which can be written in the form

$$2T = \sum_{pp'} A_{pp'} \dot{\alpha}_p \dot{\alpha}_{p'} + 2 \sum_{pj} B_{pj} \dot{\alpha}_p \dot{q}_j + \sum_{jj'} C_{jj'} \dot{q}_j \dot{q}_{j'}, \tag{3}$$

where

$$\left.\begin{aligned}
A_{pp'} &= A_{pp'}{}^{o} + \textstyle\sum_j A_{pp'j} q_j + \sum_{jj'} A_{pp'jj'} q_j q_{j'}, \\
B_{pj} &= B_{pj}{}^{o} + \sum_{j'} B_{pjj'} q_{j'}, \\
C_{jj'} &= \sum_k m_k f_{kj} f_{kj'}, \\
A_{pp'}{}^{o} &= \sum_k m_k \frac{\partial F_k}{\partial \alpha_p} \frac{\partial F_k}{\partial \alpha_{p'}}, \\
A_{pp'j} &= \sum_k m_k \left(\frac{\partial F_k}{\partial \alpha_p} \frac{\partial f_{kj}}{\partial \alpha_{p'}} + \frac{\partial F_k}{\partial \alpha_{p'}} \frac{\partial f_{kj}}{\partial \alpha_p} \right), \\
A_{pp'jj'} &= \sum_k m_k \frac{\partial f_{kj}}{\partial \alpha_p} \frac{\partial f_{kj'}}{\partial \alpha_{p'}}, \\
B_{pj}{}^{o} &= \sum_k m_k \frac{\partial F_k}{\partial \alpha_p} f_{kj},
\end{aligned}\right\} \tag{4}$$

and

$$B_{pjj'} = \sum_k m_k \frac{\partial f_{kj'}}{\partial \alpha_p} f_{kj}.$$

Equation (3) can be written in matrix form as

$$2T = \dot{\alpha}_t A \dot{\alpha} + \dot{\alpha}_t B \dot{q} + \dot{q}_t C \dot{q}, \tag{3a}$$

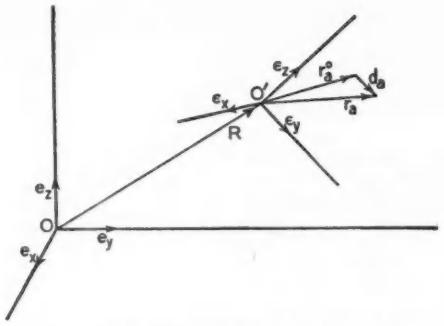

FIG. 1. Nuclear coordinates of the ath atom of a polyatomic molecule.

where $\dot{\alpha}$ and $\dot{q}$ are column matrices and the subscript t is used to indicate the transpose.

The potential energy

$$U(x_k) = U(F_k + \sum_j f_{kj} q_j)$$

may be expanded in a Taylor series about the equilibrium configuration; thus

$$2U = \sum_{kk'jj'} \left(\frac{\partial^2 U}{\partial x_k \partial x_{k'}} \right)_{x=F} f_{kj} f_{k'j'} q_j q_{j'} = \sum_{jj'} b_{jj'} q_j q_{j'}, \quad (5)$$

where

$$b_{jj'} = \sum_{kk'} \left(\frac{\partial^2 U}{\partial x_k \partial x_{k'}} \right)_{x=F} f_{kj} f_{k'j'}. \quad (6)$$

In matrix form Eq. (5) becomes

$$2U = q_t b q. \quad (5a)$$

In the equation for the kinetic energy, the first group of terms represents the energy of the system made rigid (q_j = constant) and includes the energy of translation and rotation; the third sum represents the kinetic energy of vibration (α_p = constant); the second represents the possible interactions between vibration and the other types of motion. It is desirable to set the equations in such a way that the interaction terms will be small enough to be considered as a perturbation; or in other words, that B_{pj} be small. From Eq. (4), since the q_j are small, we can insure the smallness of B_{pj} by setting

$$B_{pj}^{\circ} = 0, \quad (7)$$

which was first formulated by Casimir[5] and, as we will see later, gives the so-called Eckart conditions.

NORMAL MOLECULES

A molecule is called normal when there are no gross internal motions and when in the vibrationless state it behaves like a rigid body, (zero-point energy not being considered). If the molecule is nonlinear, there are six $\alpha_p (M=6)$, the six variables describing the classical motion of a rigid body. These are the three coordinates of a fixed point (ordinarily the center of mass) and three variables describing the orientation of a body-fixed system of axes with respect to the space-fixed axes. These are usually taken to be the Eulerian angles or the angles describing three infinitesimal rotations.

Let the index k be split into two indices i and a, where $i = x, y, z$, and $a = 1, 2, \cdots N$, N being the number of atoms in the molecule. Thus x_{ia} is the ith coordinate of the ath atom in the space-fixed system. The following nomenclature will be used (see Fig. 1): $\mathbf{R}$: position vector of the origin of the moving system O' with respect to the space-fixed origin O, $\mathbf{r}_a$: position vector of the ath atom with respect to O', $\mathbf{r}_a^{\circ}$: position vector of the ath atom in the equilibrium configuration, $\mathbf{d}_a = \mathbf{r}_a - \mathbf{r}_a^{\circ}$: displacement vector of the ath atom, $\mathbf{e}_i$: unit vector along the ith space-fixed axis, $\boldsymbol{\varepsilon}_i$: unit vector along the ith moving axis.

Since the displacements are linearly dependent on the q_j, we can write

$$\mathbf{d}_a = \sum_j \mathbf{d}_{aj} q_j, \quad (8)$$

and

$$\mathbf{r}_a = \mathbf{r}_a^{\circ} + \sum_j \mathbf{d}_{aj} q_j,$$

so that

$$x_{ia} = (\mathbf{R} + \mathbf{r}_a) \cdot \mathbf{e}_i = (\mathbf{R} + \mathbf{r}_a^{\circ}) \cdot \mathbf{e}_i + \sum_j \mathbf{d}_{aj} \cdot \mathbf{e}_i q_j; \quad (9)$$

and from Eqs. (2) and (9) we obtain

$$\left. \begin{aligned} F_{ia} &= (\mathbf{R} + \mathbf{r}_a^{\circ}) \cdot \mathbf{e}_i \\ \text{and} \qquad f_{iaj} &= \mathbf{d}_{aj} \cdot \mathbf{e}_i. \end{aligned} \right\} \quad (10)$$

By use of Eqs. (10), the elements of A, B, and

[5] H. B. G. Casimir, *Rotation of a Rigid Body in Quantum Mechanics* (J. B. Wolters, The Hague, 1931).

C given by Eqs. (4) become

$$\left.\begin{aligned}
A_{pp'}{}^{o} &= \sum_a m_a \frac{\partial(\mathbf{R}+\mathbf{r}_a{}^{o})}{\partial\alpha_p}\cdot\frac{\partial(\mathbf{R}+\mathbf{r}_a{}^{o})}{\partial\alpha_{p'}},\\
A_{pp'j} &= \sum_a m_a\left(\frac{\partial(\mathbf{R}+\mathbf{r}_a{}^{o})}{\partial\alpha_p}\cdot\frac{\partial\mathbf{d}_{aj}}{\partial\alpha_{p'}}\right.\\
&\qquad\left.+\frac{\partial(\mathbf{R}+\mathbf{r}_a{}^{o})}{\partial\alpha_{p'}}\cdot\frac{\partial\mathbf{d}_{aj}}{\partial\alpha_p}\right),\\
A_{pp'jj'} &= \sum_a m_a \frac{\partial\mathbf{d}_{aj}}{\partial\alpha_p}\cdot\frac{\partial\mathbf{d}_{aj'}}{\partial\alpha_{p'}},\\
B_{pj}{}^{o} &= \sum_a m_a \frac{\partial(\mathbf{R}+\mathbf{r}_a{}^{o})}{\partial\alpha_p}\cdot\mathbf{d}_{aj},\\
B_{pjj'} &= \sum_a m_a \frac{\partial\mathbf{d}_{aj}}{\partial\alpha_p}\cdot\mathbf{d}_{aj'},
\end{aligned}\right\} \tag{11}$$

and

$$C_{jj'} = \sum_a m_a \mathbf{d}_{aj}\cdot\mathbf{d}_{aj'}.$$

Let us obtain now the derivatives with respect to α_p which appear in Eqs. (11). For α_p corresponding to a translation,

$$\left.\begin{aligned}
&\frac{\partial\mathbf{R}}{\partial\alpha_p} = \mathbf{e}_p\\
&\text{and}\\
&\frac{\partial\mathbf{r}_a{}^{o}}{\partial\alpha_p} = \frac{\partial\mathbf{d}_{aj}}{\partial\alpha_p} = 0;
\end{aligned}\right\} \tag{12}$$

and for α_p corresponding to a rotation,

$$\left.\begin{aligned}
&\frac{\partial\mathbf{R}}{\partial\alpha_p} = 0,\\
&\frac{\partial\mathbf{r}_a{}^{o}}{\partial\alpha_p} = \mathbf{1}_p\times\mathbf{r}_a{}^{o}\\
&\text{and}\\
&\frac{\partial\mathbf{d}_{aj}}{\partial\alpha_p} = \mathbf{1}_p\times\mathbf{d}_{aj},
\end{aligned}\right\} \tag{13}$$

where $\mathbf{1}_p$ is a unit vector along the axis of rotation corresponding to α_p. If α_p are taken to be infinitesimal rotations about the moving axes,

$$\mathbf{1}_p = \boldsymbol{\varepsilon}_p. \tag{13a}$$

Let us write explicitly Eqs. (7) by taking $B_{pj}{}^{o}$ from Eqs. (11). For α_p corresponding to a translation, by use of Eqs. (12),

$$B_{pj}{}^{o} = \sum_a m_a \frac{\partial(\mathbf{R}+\mathbf{r}_a{}^{o})}{\partial\alpha_p}\cdot\mathbf{d}_{aj} = \mathbf{e}_p\cdot\sum_a m_a\mathbf{d}_{aj};$$

and Eqs. (7) require that

$$\sum_a m_a\mathbf{d}_{aj} = 0,$$

or

$$\sum_{aj} m_a\mathbf{d}_{aj}q_j = \sum_a m_a\mathbf{d}_a = 0, \tag{14}$$

which is the first Eckart condition. From the definition of $\mathbf{d}_a$, Eq. (14) is equivalent to

$$\sum_a m_a\mathbf{r}_a = \sum_a m_a\mathbf{r}_a{}^{o},$$

which means that if O' is taken at the center of mass of the molecule at the equilibrium configuration, then O' will always be at the center of mass of the molecule.

For α_p corresponding to one of the rotational coordinates (Euler angle or infinitesimal rotation), by use of Eqs. (13),

$$\begin{aligned}
B_{pj}{}^{o} &= \sum m_a \frac{\partial\mathbf{r}_a{}^{o}}{\partial\alpha_p}\cdot\mathbf{d}_{aj} = \sum_a m_a(\mathbf{1}_p\times\mathbf{r}_a{}^{o})\cdot\mathbf{d}_{aj}\\
&= \mathbf{1}_p\cdot\sum_a m_a\mathbf{r}_a{}^{o}\times\mathbf{d}_{aj},
\end{aligned}$$

and Eqs. (7) require that

$$\sum_a m_a\mathbf{r}_a{}^{o}\times\mathbf{d}_{aj} = 0,$$

or

$$\sum_{aj} m_a\mathbf{r}_a{}^{o}\times\mathbf{d}_{aj}q_j = \sum_a m_a\mathbf{r}_a{}^{o}\times\mathbf{d}_a = 0, \tag{15}$$

which is the second Eckart condition. The two Eckart conditions define the moving system. We have to describe the vibrations of the atoms in this particular system to insure the smallness of the B_{pj}.

By using Eqs. (12)–(15), the expressions (11) can be further simplified. For α_p and $\alpha_{p'}$ corresponding to translation,

$$A_{pp'}{}^{o} = \sum_a m_a\mathbf{e}_p\cdot\mathbf{e}_r = \delta_{pr}\sum_a m_a,$$

where

$$\delta_{pr}=\begin{cases}1 & \text{if } p=r,\\ 0 & \text{if } p\neq r,\end{cases}$$

and

$$A_{pp'j}=A_{pp'jj'}=0.$$

For α_p corresponding to translation and $\alpha_{p'}$ corresponding to rotation,

$$A_{pp'^{o}}=A_{pp'j}=A_{pp'jj'}=0.$$

For both α_p and $\alpha_{p'}$ corresponding to rotation,

$$\left.\begin{aligned}
A_{pp'^{o}}&=\sum_a m_a[r_a^{o}\delta_{pp'}-(\mathbf{r}_a^{o}\cdot\boldsymbol{\varepsilon}_p)(\mathbf{r}_a^{o}\cdot\boldsymbol{\varepsilon}_{p'})],\\
A_{pp'j}&=\sum_a m_a[\mathbf{r}_a^{o}\cdot\mathbf{d}_{aj}\delta_{pp'}\\
&\qquad-(\mathbf{r}_a^{o}\cdot\boldsymbol{\varepsilon}_{p'})(\mathbf{d}_{aj}\cdot\boldsymbol{\varepsilon}_p)\\
&\qquad-(\mathbf{r}_a^{o}\cdot\boldsymbol{\varepsilon}_p)(\mathbf{d}_{aj}\cdot\boldsymbol{\varepsilon}_{p'})],\\
A_{pp'jj'}&=\sum_a m_a[\mathbf{d}_{aj}\cdot\mathbf{d}_{aj'}\delta_{pp'}\\
&\qquad-(\mathbf{d}_{aj}\cdot\boldsymbol{\varepsilon}_{p'})(\mathbf{d}_{aj'}\cdot\boldsymbol{\varepsilon}_p)],
\end{aligned}\right\}\tag{16}$$

since

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})&=\mathbf{a}\cdot\mathbf{b}\times(\mathbf{c}\times\mathbf{d})\\&=\mathbf{a}\cdot[\mathbf{c}(\mathbf{b}\cdot\mathbf{d})-\mathbf{d}(\mathbf{b}\cdot\mathbf{c})]\\&=(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}).\end{aligned}$$

For α_p corresponding to translation,

$$B_{pjj'}=0;$$

and for α_p corresponding to rotation,

$$\begin{aligned}B_{pjj'}&=\sum_a m_a(\boldsymbol{\varepsilon}_p\times\mathbf{d}_{aj})\cdot\mathbf{d}_{aj'}\\&=\boldsymbol{\varepsilon}_p\cdot\sum_a m_a\mathbf{d}_{aj}\times\mathbf{d}_{aj'}.\end{aligned}\tag{17}$$

The above results show that—as we should expect—the energy of translation is separable as

$$2T_{\text{trans.}}=\mathbf{R}^2\sum_a m_a.$$

If we leave out this term in Eq. (3a), since it is of no interest in connection with band spectra,

$$2T=\tilde{\omega}A\omega+\tilde{\omega}B\dot{q}+\dot{q}_tC\dot{q},\tag{3b}$$

where ω is used for $\dot{\alpha}$ and the tilde is also used to indicate the transpose. To write the equation for the energy in Hamiltonian form, we have to define the momenta corresponding to the coordinates. Writing symbolically $\partial T/\partial\omega$ for the column matrix whose elements are the derivatives with respect to the elements of ω,

$$P=\frac{\partial T}{\partial\omega}=A\omega+B\dot{q},\tag{18}$$

or in components,

$$P_p=\sum_{p'}A_{pp'}\omega_{p'}+\sum_j B_{pj}\dot{q}_j;\tag{18a}$$

and

$$p=\partial T/\partial\dot{q}=\tilde{B}\omega+C\dot{q},\tag{19}$$

or in components,

$$p_j=\sum_p B_{pj}\omega_j+\sum_{j'}C_{jj'}\dot{q}_{j'}.\tag{19a}$$

Equation (19) gives

$$\dot{q}=C^{-1}p-C^{-1}\tilde{B}\omega,\tag{20}$$

and by substitution in Eq. (18) we obtain

$$\begin{aligned}P&=A\omega+BC^{-1}p-BC^{-1}\tilde{B}\omega\\&=(A-BC^{-1}\tilde{B})\omega+BC^{-1}p,\end{aligned}\tag{21}$$

which shows that the angular momentum contains a term, which will be designated by L, which is due to vibration alone. The effective inertial matrix will be designated by I. Thus,

$$L=BC^{-1}p,\tag{22}$$

$$I=A-BC^{-1}\tilde{B}.\tag{23}$$

In terms of P, L, and p, the velocities are

$$\left.\begin{aligned}\omega&=\mu(P-L)\\ \text{and}\qquad\dot{q}&=C^{-1}p-C^{-1}\tilde{B}\mu(P-L),\end{aligned}\right\}\tag{24}$$

where

$$\mu=I^{-1}=(A-BC^{-1}\tilde{B})^{-1}.\tag{25}$$

By use of Eq. (24) the Hamiltonian form of Eq. (3b) is now written as

$$\begin{aligned}2T&=\tilde{P}\omega+\tilde{p}\dot{q}=\tilde{P}\mu(P-L)\\&\quad+\tilde{p}C^{-1}p-\tilde{p}C^{-1}\tilde{B}\mu(P-L)\\&=(\tilde{P}-\tilde{L})\mu(P-L)+\tilde{p}C^{-1}p,\end{aligned}\tag{26}$$

which represents the final form of the Hamiltonian expression for the kinetic energy of a polyatomic molecule.

The normal coordinates are defined so that

$$\left.\begin{aligned} C_{jj'} &= \delta_{jj'} \\ \text{and} \qquad & \\ b_{jj'} &= \lambda_j \delta_{jj'}. \end{aligned}\right\} \tag{27}$$

If we consider the q_j to be the normal coordinates, Eq. (26) takes the form

$$2T = (\tilde{P} - \tilde{L})\mu(P - L) + \tilde{p}p, \tag{26a}$$

where P, L, I, and μ are given by Eqs. (19), (21), (22), (23), and (25) by taking C to be the unit matrix. Equation (26a) is given by Wilson and Howard[6] and by Herman and Shaffer[7] as a step to form the quantum-mechanical Hamiltonian.

If the atoms in the equilibrium configuration are in a straight line, only five coordinates are needed to define the rigid motion of the linear structure, namely, the coordinates of the center of mass and two angles. In consequence, there are in this case $3N-5$ degrees of vibrational freedom and therefore $3N-5$ coordinates q_j. Equation (7) gives only five conditions to define the system of moving axes and so, these will not be uniquely determined. Sayvetz[3] considers that the transformation is made as before, in which case one of the angles will enter as an arbitrary parameter. Setting this angle equal to zero, completes the definition of the moving axes.

If the molecule possesses gross internal motions which cannot be considered as small vibrations, as, for example, the ammonia and ethane molecules referred to above, Eq. (7) gives additional conditions which have to be considered separately for each case.

[6] E. B. Wilson, Jr., and J. B. Howard, J. Chem. Phys. 4, 260 (1936).
[7] R. C. Herman and W. H. Shaffer, J. Chem. Phys. 16, 453 (1948).

Reproductions of Prints, Drawings, and Paintings of Interest in the History of Physics

51. Caricature of Sir John Leslie

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(Received June 2, 1952)

An excellent caricature of Sir John Leslie, Professor of Natural Philosophy in the University of Edinburgh and inventor of the differential thermometer, was made by John Kay shortly before his death. It is here reproduced together with some amusing biographical notes.

SIR JOHN LESLIE (1766–1832), Professor of Natural Philosophy in the University of Edinburgh, is remembered nowadays primarily for his invention and application of the differential thermometer and his discoveries in the field of heat radiation. Physics teachers should think of him, however, whenever they repeat for their classes the experiment of freezing water with a vacuum pump using sulfuric acid to remove the water vapor, for it is to him that this well-known experiment was due. The other main facts of his life, together with a discussion of his principal publications, will be found in the seventh and eleventh editions of the *Encyclopaedia Britannica*, in the *Dictionary of National Biography*, and elsewhere, and so will not be repeated here.

The article in the *Dictionary of National Biography* states that a bust of SIR JOHN was executed by SAMUEL JOSEPH, of which a copy by JONN RHIND is in the National Portrait Gallery, Edinburgh, but it makes no mention of the amusing caricature sketched by JOHN KAY (1742–1826), the self-taught Scottish etcher, who from 1785 until his death produced nearly 900 plates of the oddities and celebrities of Edinburgh. Almost every notable Scotsman of the period, with the exception of Burns, was represented, and the etchings, while not entitled to high rank as works of art, possess a certain quaint originality as well as considerable fidelity as likenesses.

KAY's work was collected by HUGH PATON and published in two volumes under the title

FIG. 1. SIR JOHN LESLIE (from the caricature by John Kay).

A Series of Original Portraits and Caricature Etchings, by the late John Kay, minature painter, Edinburgh; with biographical sketches and illustrative anecdotes (Edinburgh, 1837 and 1838). The curious biographical notes that accompany the etchings were started by KAY, but were completed after his death by JAMES PATERSON, author of *The History of the County of Ayr*, aided by DAVID LAING, ALEXANDER SMELLIE, and other antiquarians.

Several pages of these biographical notes accompany the plate which is here reproduced. They read in part as follows:

"The character of Sir John has been subject to some little stricture. All have admired the inventive fertility of his genius—his extensive knowledge and vigorous mind. As a writer, however, his style has been criticised; and he has been accused as somewhat illiberal in his estimate of kindred merit, while he is represented to have been credulous in matters of common life, and sceptical in science. 'His faults,' says his biographer, 'were far more than compensated by his many good qualities—by his constant equanimity, his cheerfulness, his simplicity of character, almost infantine, his straight-forwardness, his perfect freedom from affectation' and, above all, his unconquerable good nature. He was, indeed, one of the most placable of human beings; and if, as has been thought, he generally had a steady eye, in his wordly course, to his own interest, it cannot be denied that he was, notwithstanding, a warm and good friend, and a relation on whose affectionate assistance a firm reliance could ever be placed.' In this character we are disposed to concur. One slight blemish, however, has been overlooked—personal vanity; for, strange to say, although in the eyes of others the worthy knight was very far from an Adonis, yet in his own estimation he was a perfect model of male beauty.

"The general appearance of Sir John is well represented in the Print which precedes this notice. He was short and corpulent—of a florid complexion*—and his front teeth projected considerably. In later life his corpulence increased;† he walked with difficulty; and he became rather slovenly in his mode of dress—a circumstance the more surprising, as his anxiety to be thought young and engaging continued undiminished."

* "What the natural color of his hair may have been we cannot say; but in consequence of the use of some tincture —Tyrian dye it is said—it generally appeared somewhat of a purple hue."

† "When unbending his mind from severer labours, the knight resorted to Apicius; and to his success in reducing to practice the gastronomical propositions of that interesting writer has been ascribed his somewhat remarkable exuberance of abdomen. A legal friend, now, alas! no more, once witnessed an amicable contest between Sir John and an eminent individual, celebrated for his taste in *re culinaria*. The latter was invincible in the turtle soup and cold punch, but the former carried all before him when the 'sweets' were placed on the table. To show how easily the victory was won, besides other fruits produced with the dessert, the knight, without any effort, devoured nearly a couple of pounds of almonds and raisins."

Demonstrations to Provide Data for Student Computation in Electricity*

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(Received March 24, 1952)

Two panel boards were devised for use in obtaining data for student computations in electricity, the first for study of Ohm's law and Kirchhoff's laws applied to networks, and the second for study of line loss in power transmission. The boards are effective teaching aids because of their good visibility and the ease with which circuits can be altered. The data are sufficiently precise to provide for results in agreement with physical laws.

DURING the past two years our staff has developed demonstrations to provide data for computations in general physics for engineering students. These computations parallel the content of the classwork and alternate with the usual laboratory experiments during the two-year physics course. The staff has found that computations based on data from experimental investigations carried on by the group result in significant learning that is not obtained through the solution of problems prepared by the instructor.

Apparatus available from scientific houses was not sufficiently large to enable groups of thirty students to observe the procedures, hence it was necessary to plan and build the equipment. Two of these assemblies will be described. The first, a Vari-circuit board, was used to provide data for computations involving Ohm's law, equivalent resistance, and Kirchhoff's laws; the second was for computing line loss in power transmission.

VARI-CIRCUIT BOARD

A section of five-eighth inch plywood 4×3 ft was supported in a vertical plane by two short slotted feet. Visibility was increased by painting the board yellow, and the circuit was made with No. 14 black insulated stranded wire stapled to the board. (See Fig. 1). Nuway snap connectors[1] were used as terminals for resistors and meters. These connectors provided a large degree of flexibility and convenience as meters and can be connected to any two terminals by merely pressing the connectors on the meter leads onto the terminals. Thus, in measuring potential drops satisfactory connections can be made without disturbing the circuit. Additional components may be quickly inserted in the circuit by pulling off leads and snapping the new leads on the terminals. The connectors do not become loose with long use and continue to provide good contact with insignificant resistance.

Resistor blocks were built by mounting low cost radio resistors and snap connector terminals on blocks or wood 2×4×$\frac{3}{4}$ in. The resistance of each resistor was measured with a high precision Wheatstone bridge and the values stenciled on the edge of the blocks. The blocks were suspended on the board by two round-headed screws which fitted loosely into holes drilled through the board. Thus, it was possible to move or replace

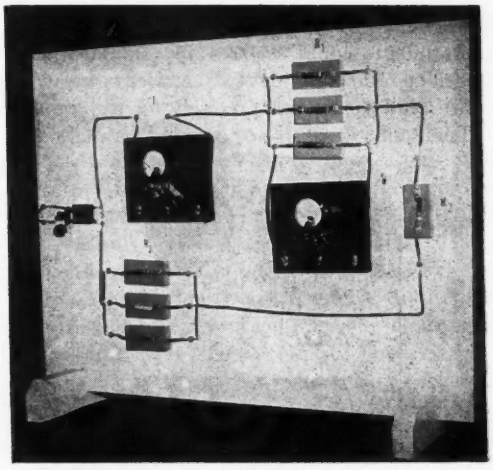

FIG. 1. Vari-circuit board. (Meters shown were replaced with 7-in. wide view meters.)

* Based upon a paper read at the twenty-first annual meeting of the AAPT, Barnard College, New York, January 31, 1952.

[1] Nuway Snap Connectors may be purchased from Hathaway Manufacturing Company, 4347 West Twenty-fourth Street, New York, New York.

TABLE I. Sample computation. To find the total resistance of a circuit by (1) applying Ohm's law and (2) by computing the equivalent resistance.

R	V (volts)	I (amperes)	$\frac{V}{I}=R$ (ohms)	R Standard (ohms)
R_{1A}	1.72	0.216	7.96	8.00
R_{1B}	1.72	0.536	3.21	3.19
R_{1C}	1.72	0.574	3.00	2.99
R_1 (Exp)	1.72	1.33	1.29	
R_1 (Cal)			1.30	
R_2	5.50	1.26	4.36	4.32
R_{3A}	2.10	0.650	3.23	3.21
R_{3B}	2.10	0.420	5.00	4.96
R_{3C}	2.10	0.200	10.5	10.6
R_3 (Exp)	2.10	1.26	1.67	
R_3 (Cal)			1.65	

$R_T=1.29+4.35+1.67=7.31$ ohms

resistors easily and rapidly. Meters were mounted on larger blocks and suspended on the board in a similar manner. This made changes in meter positions convenient and the meters were available for other experiments. Seven-inch wide view meters were used. During the operation all the students could see the approximate needle positions and one or two students near the board read the meters precisely for the group.

TABLE II. Sample computation. To compute the total resistance from measurements of the potential drop and current for the complete circuit.

V_1 (volts)	V_2 (volts)	V_3 (volts)	E $(E=V_1+V_2+V_3)$ (volts)	E Measured (volts)
1.72	5.50	2.10	9.32	9.36

$R_T=E/I=\frac{9.36}{1.26}=7.44$ ohms. Percent difference for the two values of $R_T=1.7$.

The board is set up for a series-parallel circuit

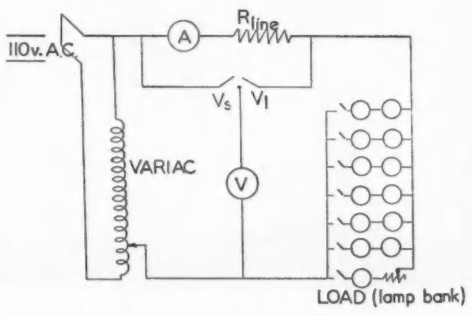

FIG. 2. Power transmission board circuit.

in Fig. 1. By changing the snap leads at R_1 and R_3 all the resistances can be connected in series within a few seconds. By changing other connections various series-parallel combinations may be produced, and resistor blocks may be interchanged or replaced with ones of different values. Resistor R_2 is frequently replaced with a battery; thus, a second emf is introduced into the circuit for computations based on Kirchhoff's laws. The variable resistor at the source S was used to vary the emf of the supply and data were taken for several current values. Quite complicated networks may be set up rapidly by snapping additional leads, with or without resistances, across various points on the board.

A sample set of data and computations for the series-parallel circuit of Fig. 1 is included below. Students computed the resistance of each resistor from measurements of potential drops and currents. The total resistance R_T found by calculating the equivalent resistance was in close agreement with the value computed by using the potential drop across the entire circuit and the current at I. The measured emf was found to equal the sum of the potential drops within reasonable experimental error. (See Tables I and II.)

TABLE III. A sample student computation.

V_1 (Load) (volts)	(amps)	P_L (Load) $V_L \times I$ (watts)	V_S (Source) (volts)	P_S (Source) $V_S \times I$ (watts)	P (Loss) P_S-P_L (watts)	Percentage Loss	Percentage Efficiency
50	1.6	80	52.3	83.7	3.7	4.0	96.0
50	3.2	160	54.6	175	14.8	8.4	91.6
50	4.8	240	57.0	274	33.6	12.2	87.8
50	6.4	320	60.0	384	64.0	16.7	83.3
50	8.0	400	63.0	504	104	20.6	79.4
100	0.8	80	101	81.0	0.96	1.1	98.9
100	1.6	160	102	164	3.8	2.2	97.8
100	2.4	240	104	248	8.4	3.4	96.6
100	3.2	320	105	334	14.4	4.5	95.5
100	4.0	400	106	424	24.0	5.6	94.3

LINE LOSS IN POWER TRANSMISSION

The purpose of this demonstration was to indicate that the efficiency with which power is transmitted increases with increase in voltage. This means that the power loss for the same load increases as the voltage decreases.

The apparatus consisted of a Variac, ammeter, voltmeter, variable load (1 lamp-blank) and a simulated line resistance, a coil of approximately

2 ohms rated at 100 watts. All of these items except the Variac were mounted on a panel to afford better visibility to a large number of students. The circuit is shown in Fig. 2.

The procedure followed was to maintain the output voltage at a constant value of 50 volts and to increase gradually the current through the load. Currents of 1.6, 3.2, 4.8, 6.4, and 8 amperes were obtained by carefully adjusting the Variac and the lamp load. At these values the power delivered to the load was 80, 160, 240, 320, and 400 watts. Also, at each setting of the Variac the potential drop across the Variac was taken.

This procedure was again followed using 100 volts as the constant potential across the load. However, this time the current readings were halved, and, therefore, the power delivered to the load remained at the same values. The corresponding voltage drops across the input were recorded.

From the above data the students calculated the power input, ($V_S \times I$ = watts) at the source, and the power delivered ($V_L \times I$ = watts) to the load. The difference of these two values was the power loss in the transmission lines. The percent loss in the line and the efficiency of the transmission were calculated also. (See Table III.)

To permit a better understanding of these figures and a clearer picture of the conclusions to be reached, two pairs of curves were plotted.

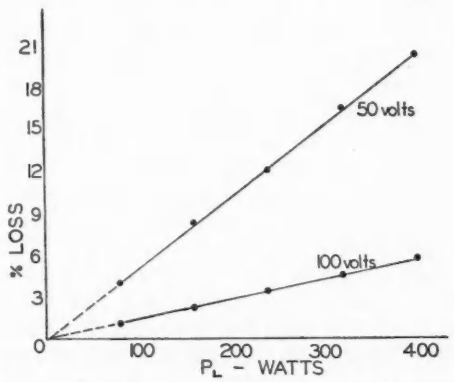

FIG. 3. Percent line loss as a function of power at the load.

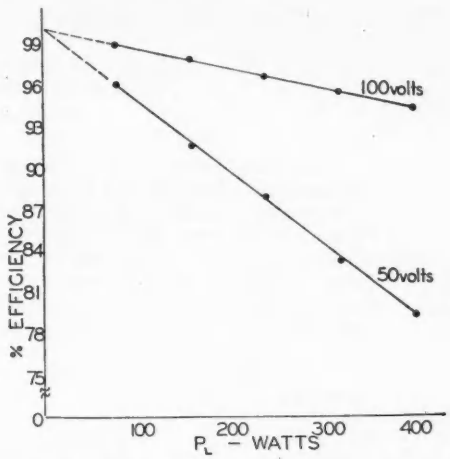

FIG. 4. Percent efficiency as a function of power at the load.

In Fig. 3, a curve representing percent loss *versus* power at the load was drawn when the potential drop across the load was maintained at 50 volts, and a similar curve for 100 volts was drawn. Similarly, percentage efficiency *versus* power at the load was also drawn and is shown in Fig. 4. The graphs clearly indicate the advantage of transmitting power at a high voltage since the efficiency increases and the loss decreases with increased voltage.

The two demonstrations described have been used in freshmen physics classes during the past two years. We have found that the students became intensely interested in the problems proposed by the demonstrations, and they understood the significance of the computations as they applied them to the operating apparatus. Principles and laws became meaningful to students as they used them in dealing with concrete situations. The teachers have found the apparatus convenient to operate and useful as a teaching aid because of its visibility and the degree of precision of the data obtained.

The authors wish to acknowledge the helpful suggestions and advice of Paul O. Hoffmann, Chairman of the Technological Division and Physics Department at Newark College of Engineering.

The Closed Differential Pulley: Analysis of Its Rotational and Linear Motions*

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(Received February 25, 1952)

A mechanical system of special design, primarily for instructional purposes, is proposed, mathematically analyzed, and tested experimentally. It may be regarded as a hybrid form between the compound wheel-and-axle and the differential pulley or chain hoist. Although the design is simple, the analysis is not too obvious at the start, and is somewhat complicated, exhibiting coordinated angular and linear motions with constant accelerations due entirely to the weight of the movable pulley. While the analysis assumes no friction, the actual heat loss can be readily calculated from time and displacement measurements and the equivalent friction force determined. Numerous equations and numerical data are included. The system offers various challenges to student interest in classical mechanics, both on the lower-division and advanced levels. Particular experimental work is suggested and analytical exercises given for use by teachers.

ROTATIONAL mechanics, aside from any lack of attention on programs, is easily underemphasized in the teaching of general physics for two reasons. A proportionately large amount of the student's time at the start is needed in the mechanics of linear motion, such as of falling bodies, including projectiles. Then, having learned the basic formulas of translatory motion, he finds it simple to apply the *same algebra* to the corresponding problems of rotation, and to get the right answers without spending nearly the time thinking about the *physical* situations as was spent with the problems in translation.

PULLEY SYSTEMS FOR STUDYING ROTATIONAL AND LINEAR MOTIONS UNDER NON-EQUILIBRIUM CONDITIONS

Pulley systems are dealt with in general physics as cases of equilibrium, the term "mechanical advantage" being tacitly so defined. An exception occurs in the Atwood machine, with its single pulley and unbalanced weights giving acceleration to the system of masses. It appeared challenging to study a more complex pulley system in which unbalanced forces act to produce accelerations.

The older textbook of physics edited by Duff[1] gave a problem in a nonequilibrium system of three coplanar pulleys, as follows: "A cord passes over two fixed pulleys and through a third pulley suspended between them. A mass of 10 grams is attached to one end of the cord, a mass of 5 grams to the other end, and the suspended pulley and the attached weights weigh 2 grams. The cords being all vertical, what are the accelerations of the three masses?" According to the answers given in the book (809, 639, and 724 cm/sec^2, the outside weights moving downward, the movable pulley upward), the solution assumed no friction, and also neglected moments of inertia. Thus, the tension in the cord could be regarded as everywhere the same, simplifying the problem. In such a situation the vertical linear acceleration of the movable pulley is the average of the accelerations of the two outside weights, with algebraic signs reversed.

THE "CLOSED" DIFFERENTIAL PULLEY

We now propose to consider a particular *modification* of one of the more complex classical pulley systems. This modification is called the "closed" differential pulley, illustrated in Fig. 1. It may be regarded as a hybrid between the compound wheel-and-axle and the chain hoist or differential pulley. Also, it may be thought of either (1) as a compound wheel-and-axle with the wheel removed, or (2) as a differential pulley in which the characteristic free, or "open," loop is missing, which suggested the term "closed." In case (1) the wheel might be retained but without any torque acting on it, the wheel then merely con-

* Paper presented at the meeting of the American Association of Physics Teachers, Southern California Section, held at Pomona College, April 21, 1951.

[1] A. W. Duff, *A Text-Book of Physics* (Blakiston, 1916), fourth revised edition, p. 165, problem 18. This problem does not appear in the sixth revised edition, 1926, with the changed title, *Physics for Students of Science and Engineering*, nor, I believe, in any subsequent editions.

tributing to the total moment of inertia of a rotating rigid body. A glance at Fig. 1 shows how the device works.

If an actuating torque from outside were applied in the present system, it would be of the ordinary differential-pulley type. But here there is no torque from outside. The operation is therefore due entirely to the weight of the movable pulley and any additional mass attached to it.

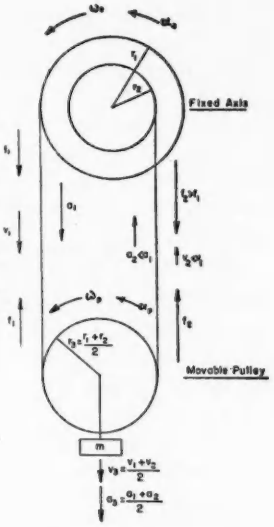

FIG. 1. The closed differential pulley. Certain vector quantities, the velocities and accelerations both angular and linear, are indicated, but not displacements.

THE ANALYSIS

In its mechanical design the apparatus is quite simple, rather disarmingly so when one starts analyzing. Eventually, however, one finds a minimum of twenty-four different magnitudes in the equations developed. In addition to these twenty-four there are others, related to one or more of the twenty-four. Although analytically complicated, the mechanical system is adaptable to the laboratory in general physics, with the analysis abbreviated. For the more technical pre-engineering physics, there can be less abbreviation. For a still more detailed study, it might be used in advanced classical mechanics.

ASSUMPTIONS

While no apparatus is free from friction, it is acceptable for analysis to assume this freedom, as is often done. Seemingly inconsistent with the supposed absence of friction is the further assumption that no slippage of the one string occurs anywhere. This condition is met experimentally at the axle by having the two ends of the string attached permanently to it, and at the pulley by admitting enough friction to prevent slippage. For analysis, it can be imagined that the string is a frictionless chain, the links of which engage cogs on the pulley, as in the chain hoist.

In the present case we further "idealize" by assuming that the moving elements all lie in the same plane. The experimental justification for this assumption of coplanarity in getting usable "working equations" will appear later.

The principal assumptions are: (1) coplanarity of all the moving elements; (2) no friction, except at rim of pulley; (3) no slippage of string; and (4) thickness and weight of string negligible.

QUANTITIES APPEARING

For brevity, the term "axle" will be used instead of "compound axle." Subscripts a and p refer to axle and pulley, respectively. Subscripts 3 and p mean the same thing, both referring to the movable pulley. We have: t, time; r_1, r_2, radii of large and small drums of axle; m, mass of pulley and any attached weights; I_a, I_p, moments of inertia; g, gravity acceleration; f_1, f_2, tensions in cord at left and right (Fig. 1); τ_1, τ_2, torques applied by cord at left and right, to large and small drums of axle; τ_a, net torque on axle; τ_p, torque on pulley; a_1, a_2, linear accelerations of left and right cords; a_3, linear acceleration downward of pulley unit; α_a, α_p, angular accelerations; h_1, h_2, h_3, linear displacements, with respect to the "ground," of left cord, right cord, and pulley unit, in time t, time being counted from instant of release of system; v_1, v_2, v_3, corresponding linear velocities with respect to the ground, attained in time t; ω_a, ω_p, angular velocities; ϕ_a, ϕ_p, angular displacements; ν_a, ν_p, T_a, T_p, instantaneous frequencies and periods; and finally n_a, n_p, total revolutions of axle and pulley—all in time t.

GROUPING OF THE QUANTITIES

The algebraic quantities total 24 as a minimum, as follows: the independent variable t; 6

independent parameters, r_1, r_2, I_a, I_p, g, m; 9 dependent parameters (dependent on the independent parameters but not on the time), f_1, f_2, τ_1, τ_2, (τ_a), τ_3, a_1, a_2, a_3, α_a, (α_p); and 8 variables changing with time, h_1, h_2, h_3, v_1, v_2, v_3, ω_a, $(\omega_p, \nu_a, \nu_p, T_a, T_p)$, ϕ_a, (ϕ_p, n_a, n_p). Adding to the 24 the quantities in parentheses, we have an over-all total of 34 algebraic symbols.

OUTLINE OF THE DERIVATIONS

With the beginning student in mind, we give a resumé of the mathematical analysis. The particular algebraic route followed is probably variable, and might conceivably be shortened. While derivational steps will be interspersed with discussion, the results will later be summarized. Downward direction is positive.

RADII AND LINEAR DISPLACEMENTS

Let us assume that

$$r_3=(r_1+r_2)/2. \tag{1}$$

Then since the linear displacements h_1, h_2 of the cord at left and right are equal to the lengths of arc traversed by any fixed points on the peripheries of the two drums, and since these are coaxial and fastened together, the corresponding angle turned through by the axle is

$$\phi_a=h_1/r_1=-h_2/r_2, \text{ whence } h_1/h_2=-r_1/r_2, \tag{2}$$

h_2 being negative because the cord on the right moves upward. This is a relation among the independent parameters r_1, r_2, which are independent of the time, and the variables h_1, h_2, dependent on the time.

Now h_1 is the length of string that in time t unwraps from the large drum, while h_2 is the length which in the same time wraps up on the small drum. Remembering that h_2 is negative, we have h_1+h_2 as the total additional length of "free" string in time t. The free string consists at any instant of the two vertical segments not in contact with either axle or pulley. These remain equal, although each increases in length as the pulley descends, so that the additional free length is divided equally between the cord at the left and that at the right (Fig. 2). Therefore, the pulley must have descended a distance

$$h_3=(h_1+h_2)/2. \tag{3}$$

This is a geometrical and mechanical fact we can accept. Substituting in Eq. (3) from (2) gives

$$h_3=ch_1/(2r_1)=-ch_2/(2r_2), \tag{4}$$

introducing for brevity the symbol $c=r_1-r_2$.

LINEAR VELOCITIES AND ACCELERATIONS

It is unnecessary to resort to calculus, any more than in problems of freely falling bodies where acceleration is assumed constant. However, we shall use some calculus. Differentiation of Eq. (3) with respect to time gives

$$v_3=(v_1+v_2)/2; \tag{5}$$

similarly from Eq. (5),

$$a_3=(a_1+a_2)/2. \tag{6}$$

Differentiating Eqs. (2) and (4),

$$v_1/v_2=h_1/h_2=-r_1/r_2, \tag{7}$$

and

$$v_1/v_3=h_1/h_3=2r_1/c, \tag{8}$$

whence by Eqs. (7) and (8), or by Eq. (4),

$$v_2/v_3=h_2/h_3=-2r_2/c. \tag{9}$$

Further, differentiation of Eqs. (7) and (8) gives

$$a_1/a_2=v_1/v_2=h_1/h_2=-r_1/r_2, \tag{10}$$

and

$$a_1/a_3=v_1/v_3=h_1/h_3=2r_1/c, \tag{11}$$

whence from Eqs. (10) and (11)

$$a_2/a_3=v_2/v_3=h_2/h_3=-2r_2/c. \tag{12}$$

Thus the *ratios* of the v's and the h's are the same as those of the a's, all taken in corresponding pairs; but whereas the a's are constant, both the v's and the h's are dependent on time, the v's varying as the first power of t, the h's as its square, as in ordinary free fall.

ANGULAR VELOCITIES AND ACCELERATIONS, AND TORQUES

Since the axis of the compound axle remains stationary, the v's being referred to the "ground," we have

$$\omega_a=v_1/r_1=-v_2/r_2, \tag{13}$$

which is the same relation among v_1, v_2, r_1, r_2

as Eq. (7), while

$$\alpha_a = a_1/r_1 = -a_2/r_2, \tag{14}$$

agreeing similarly with Eq. (10).

Expressions are now wanted for the corresponding angular velocity and angular acceleration of the pulley, which both rotates and has translatory motion as a whole. Here we must look at the physical picture closely. Referring to Fig. 2, we see that point F on the descending string at the left, which at any instant during the motion has just made contact with the pulley, will in any time t thereafter have moved from its initial position, measured along the string in its instantaneous position at time t, a distance h_1. However, in this same time the pulley center has descended a distance h_3, so that the length of string that has made contact with the pulley rim in this time is $h_1 - h_3 = s$, the length of arc through which any point on the perimeter must have moved. Therefore, the pulley's angular displacement is

$$\phi_p = s/r_3 = (h_1 - h_3)/r_3. \tag{15}$$

As before, differentiation gives

$$\omega_p = (v_1 - v_3)/r_3, \tag{16}$$

while Eq. (16) yields

$$\alpha_p = (a_1 - a_3)/r_3. \tag{17}$$

The net torque acting on the axle is

$$\tau_a = \tau_1 + \tau_2 = f_1 r_1 - f_2 r_2 = I_a \alpha_a. \tag{18}$$

The component torque τ_2 on the small drum is negative, or clockwise. But the cord tensions, f_1 and f_2, are both regarded throughout the analysis as positive. The torque on the pulley must be

$$\tau_p = (f_2 - f_1) r_3 = I_p \alpha_p. \tag{19}$$

The pulley accelerates counterclockwise, and the torque has that direction. This means that $f_2 > f_1$, although from Eq. (18), $(f_2 r_2) < (f_1 r_1)$.

AN IMPORTANT PRINCIPLE IN ROTATIONAL MECHANICS

Next, connecting the linear acceleration a_3 to the weight mg, we meet a desirably simple situation where the working of an important principle in the mechanics of rigid bodies is illustrated. The situation is simple because the forces, only three in number, are not only coplanar but also parallel. The principle states that "all the forces applied to a rigid body may be replaced by a single force and a single couple."[2]

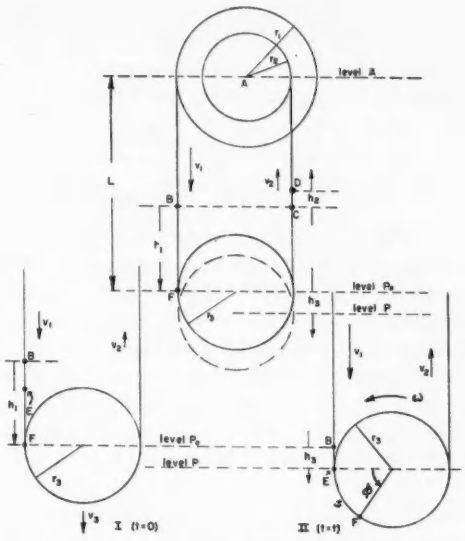

FIG. 2. Spread diagram as an aid in analyzing the performance of the closed differential pulley. Diagram shows the relation $h_1 = h_2 + s$, where $s/r_3 = \phi$, the angular displacement of pulley. Not to exact scale throughout: actually, $|h_3| < |h_2|$.

Suppose that the beginning student reasons wrongly in the following manner: Of the total linear force, $f_1 + f_2$, applied through the string and acting upward on the pulley, Fig. 1, the amount $(f_2 - f_1)$ is "used up" in producing, according to Eq. (19), angular acceleration of the pulley. Therefore, there is left, as the total externally applied force opposing the weight mg, the amount $(f_1 + f_2) - (f_2 - f_1) = 2f_1$, so that the equation for the linear acceleration of the pulley unit would in that event be $mg - 2f_1 = ma_3$. But if the student proceeds on this wrong basis, he will not find the resulting equations agree with the principle of the conservation of energy, $mgh_3 = \frac{1}{2} I_a \omega_a^2 + \frac{1}{2} I_p \omega_p^2 + \frac{1}{2} m v_3^2$. But applying the principle above to get the resultant force on the pulley, we subtract from mg the sum $(f_1 + f_2)$, while the resultant couple is τ_p as in Eq. (19).

[2] A. G. Webster, *The Dynamics of Particles and of Rigid, Elastic and Fluid Bodies* (B. G. Teubner, Leipzig, 1912), second edition, p. 209.

The results will now agree with the energy equation. No part of (f_1+f_2) is "used up" in producing angular acceleration. This important relation as it affects the pulley is then

$$mg-(f_1+f_2)=ma_3. \quad (20)$$

THE DEPENDENT PARAMETERS

By choosing from the foregoing, we seek expressions for the dependent parameters—forces, torques, accelerations—which are independent of time, in terms entirely of the independent parameters, r_1, r_2, I_a, I_p, g, and m.

Eliminating α_a from Eqs. (14) and (18) leaves, $a_1=r_1(f_1r_1-f_2r_2)/I_a$ (21); and α_p from Eqs. (17) and (19), $a_3=a_1-r_3^2(f_2-f_1)/I_p$ (22). Eliminate a_3 from Eqs. (6) and (22), $a_2=a_1-2r_3^2(f_2-f_1)/I_p$ (23); and a_2 from Eqs. (14) and (23), $a_1=r_1b(f_2-f_1)/(2I_p)$ (24) where $b=r_1+r_2$. Finally, eliminate a_1 from Eqs. (21) and (24), $f_2/f_1=(I_ab+2I_pr_1)/(I_ab+2I_pr_2)=A/B$ (25), where A and B are convenient substitution symbols, for functions of the independent parameters only. Accelerations are constant by Eqs. (21), (30), and (31), a conclusion reachable also by a physical *reductio-ad-absurdum* argument.

Equation (25) is of a symmetrical type, the only difference between numerator and denominator being in the second term, where r_1 appears in the numerator and r_2 in the denominator. Since $r_1>r_2$, we have $f_2>f_1$; that is, tension in the cord on the right is greater than that on the left. This indicates that the analysis is correct thus far, since the pulley accelerates counterclockwise.

Eliminate a_3 from Eqs. (6) and (20), $a_2=2g-a_1-2(f_1+f_2)/m$ (26); and a_2 from Eqs. (14) and (26), $a_1=2r_1[mg-(f_1+f_2)]/(mc)$ (27); then a_1 from Eqs. (24) and (27), $f_2=[(mbc-4I_p)f_1+4I_pmg]/(mbc+4I_p)$ (28).

For brevity write I for I_a+I_p, then let $Q=4I+mc^2$ (29); then eliminate f_2 from Eqs. (25) and (28), $f_1=2Bmg/(bQ)$ (30); which is the desired expression for f_1, in terms entirely of the independent parameters. From Eqs. (30) and (25), $f_2=2Amg/(bQ)$ (31), which is the desired expression for f_2.

For further brevity let $S=(r_1-r_2)mg/Q=cmg/Q$ (32). Then by Eq. (30), $f_1=2AS/(bc)$ (33); while by Eq. (31), $f_2=2BS/(bc)$ (34), where $A=(I_ab+2I_pr_1)$, and $B=(I_ab+2I_pr_2)$, as in Eq. (25). From Eqs. (30), (31), and (32), $(f_2-f_1)=4I_pS/b=2I_pS/r_3$ (35). Similarly, $f_1+f_2=4IS/c$ (36). Substitute Eq. (35) in Eq. (24), $a_1=2r_1S$ (37). From Eqs. (23), (35), and (37), $a_2=-2r_2S$ (38); and by Eqs. (6), (37), and (38), $a_3=cS$ (39). Equations (37), (38), and (39) are the desired expressions for the three linear accelerations.

From Eqs. (18), (33), and (34), $\tau_a=2I_aS$ (40); and by Eqs. (19) and (35), $\tau_p=2I_pS$ (41). The net torques on the axle and pulley are counterclockwise. Incidentally, from Eqs. (18) and (19), $\tau_a/\tau_p=I_a\alpha_a/(I_p\alpha_p)$ (42); while from Eqs. (40), (41), and (42), $\tau_a/\tau_p=I_a/I_p$ (43). By Eqs. (42) and (43), $\alpha_a=\alpha_p$, which means that axle and pulley always agree in phase. To check this result, by Eqs. (14) and (37), $\alpha_a=2S$ (44); and from Eqs. (17), (37), and (39), $\alpha_p=2S=\alpha_a=\alpha$ (45). This result is obtainable directly from Eqs. (11), (12), and (17).

Now $\tau_1=f_1r_1$, whence by Eq. (30), $\tau_1=r_1Amg/(bQ)=2r_1AS/(bc)$ (46). Also $\tau_2=f_2r_2$, whence by Eq. (31), $\tau_2=-2r_2Bmg/(bQ)=-2r_2BS/(bc)$ (47).

To check Eq. (40) by Eqs. (46) and (47), since $\tau_a=\tau_1+\tau_2$, τ_2 being negative, we get $\tau_a=2I_aS$, the same as Eq. (40). Further, by Eqs. (46) and (47), we have $\tau_1/\tau_2=-Ar_1/(Br_2)$ (48), which shows that since $A>B$, and $r_1>r_2$, $|\tau_1|>|\tau_2|$, although, as noted above, $f_1<f_2$.

Also, since by Eq. (19), $\tau_p=(f_2-f_1)r_3$, by Eq. (35), $\tau_p=2I_pS$ (49). Incidentally, by Eqs. (46) and (49), $\tau_1/\tau_p=r_1A/(I_pbc)$ (50); and by Eqs. (47) and (49), $\tau_2/\tau_p=-r_2B/(I_pbc)$ (51).

The desired expressions have thus been obtained for all the dependent parameters in terms of the independent ones.

DERIVATIONS FOR THE VARIABLES DEPENDENT ON TIME

We next want expressions, in terms of independent parameters, for the variables dependent on time. These include the velocities and displacements, both linear and angular.

(A) The Pulley Displacement h_3 as Independent Variable

The distance h_3 through which the pulley has descended is a function of time, $h_3=cSt^2/2$

(68). Because this distance is readily measurable, expressions in terms of it are desirable. Accelerations are constant.

$v_1^2=2a_1h_1$ (52); $v_2^2=2a_2h_2$ (53); $v_3^2=2a_3h_3$ (54). From Eqs. (2) and (3), $h_1=2r_1h_3/c$, (55); and similarly, $h_2=-2r_2h_3/c$ (56). By Eqs. (25), (36), and (39), $v_1=2r_1\sqrt{(2Sh_3/c)}$ (57); by Eqs. (26), (37), and (40), $v_2=-2r_2\sqrt{(2Sh_3/c)}$ (58); and by Eqs. (3), (27), and (38), $v_3=\sqrt{(2cSh_3)}$ (59). To check, note that Eq. (58) could have been got directly by Eqs. (7) and (57), and likewise Eq. (59) either by Eqs. (8) and (57), or by Eqs. (9) and (58). Equations (55) to (59) are the desired expressions for linear displacements and velocities in terms of h_3, the pulley displacement, a function of time.

Now, by Eqs. (13) and (57), $\omega_a=v_1/r_1=2(2Sh_3/c)^{\frac{1}{2}}$ (60); while by Eqs. (16), (57), and (59), $\omega_p=(v_1-v_3)/r_3=2(2Sh_3/c)^{\frac{1}{2}}=\omega_a=\omega$ (61). Since $t=0$ at the instant of the system's release, $\phi_a=\omega_a^2/(2\alpha_a)$, whence by Eqs. (44) and (60), $\phi_a=2h_3/c$ (62); and similarly by Eqs. (45) and (61), $\phi_p=2h_3/c=\phi_a=\phi$ (63). Equations (60) to (63) give in terms of h_3 angular velocities and angular displacements of axle and pulley, and they are the same for both.

In view of these results, we have for the common frequency of rotation attained in time t, $\nu=\omega/(2\pi)=(2Sh_3/c)^{\frac{1}{2}}/\pi=1/T$ (64), where T is their common period. Also, the total revolutions are the same for both pulley and axle, $n=\phi/(2\pi)=h_3/(\pi c)=h_1/(2\pi r_1)=-h_2/(2\pi r_2)$ (65).

(B) The Time t as Independent Variable

Since in general $h=at^2/2$, we have by Eqs. (37), (38), and (39), respectively, $h_1=r_1St^2$ (66); $h_2=-r_2St^2$ (67); $h_3=cSt^2/2$ (68). Also similarly, $v_1=2r_1St$ (69); $v_2=-2r_2St$ (70); $v_3=cSt$ (71). For the corresponding angular quantities ω and ϕ, since $\omega=\alpha t$ and $\phi=\alpha t^2/2$, by Eqs. (44) and (45), $\omega_a=2St=\omega_p=\omega$ (72). Similarly $\phi_a=St^2=\phi_p=\phi$ (73). Equations (66) to (73) are the desired expressions for linear and angular velocities and displacements in terms of time. Frequency $\nu=\omega/(2\pi)=St/\pi=1/T$, (74); and total revolutions $n=\phi/(2\pi)=St^2/(2\pi)$ (75).

SUMMARY OF EQUATIONS FOR THE DEPENDENT PARAMETERS

For convenient reference, in addition to sections (A) and (B) just above, we assemble "working equations" from the expressions preceding those, remembering the mechanical condition that the pulley diameter is the average of the two axle-diameters, $r_3=(r_1+r_2)/2=b/2$ (1) bis. Wherever the factor $b=(r_1+r_2)$ appears, it is easy to introduce r_3 into the equation. As to the factor $c=(r_1-r_2)$, we shall comment later under "Equivalent Axle." Conciseness, as has been seen, is served by the abbreviating functions Q and S. These are parametric in character, since they are functions entirely of the independent parameters, S being a function of Q. They also will be discussed presently. We have, $Q=4I+mc^2$, (29); $S=cmg/Q$, (32).

Cord tensions: $f_1=2AS/(bc)$ (33); $f_2=2BS/(bc)$ (34); where $A=I_ab+2I_pr_1$, and $B=I_ab+2I_pr_2$, as in Eq. (25). Linear accelerations: descending cord, $a_1=2r_1S$ (37); ascending cord, $a_2=-2r_2S$ (38); pulley unit, $a_3=cS$ (39). The torques: on large drum of axle, $\tau_1=2r_1Amg/(bQ)=2r_1AS/(bc)$ (46); on small drum, $\tau_2=-2r_2Bmg/(bQ)=-2r_2BS/(bc)$ (47); net torque on axle, $\tau_a=\tau_1+\tau_2=f_1r_1-f_2r_2=2I_aS$ (40); net torque on pulley, $\tau_p=(f_2-f_1)r_3=2I_pS$ (41). Angular acceleration common to both axle and pulley, $\alpha=2S$ (45).

QUANTITIES OTHER THAN h_3 OR t AS INDEPENDENT VARIABLES

Obviously with time considered as a parameter any of the dependent parameters, themselves independent of time, can be treated as independent variables, some other dependent parameter being regarded as dependent variable. By plotting these two the locus of the equation can be studied. Two examples follow:

(*a*) *Phase angle ϕ as independent variable.* From Eq. (73), $t^2=\phi/S$, whence by Eq. (66), $h_1=r_1\phi$; by Eq. (67), $h_2=-r_2\phi$; by Eq. (68), $h_3=c\phi/2$; by Eq. (69), $v_1=2r_1\sqrt{(S\phi)}$; by Eq. (70), $v_2=-2r_2\sqrt{(S\phi)}$; by Eq. (71), $v_3=c\sqrt{(S\phi)}$; by Eq. (72), $\omega=2\sqrt{(S\phi)}$; by Eq. (74), $\nu=\sqrt{(S\phi)}/\pi$; and by Eq. (75), $n=\phi/(2\pi)$. This group of equations is collectively designated (76).

(*b*) *Total rotations n as independent variable.*

Since $\phi = 2\pi n$, we have by Eq. (73), $t^2 = 2\pi n/S$, so that substitution again in Eqs. (66) to (75) yields the obvious relations: $h_1 = 2\pi r_1 n$; $h_2 = -2\pi r_2 n$; $h_3 = \pi c n$; $v_1 = 2r_1(2\pi Sn)^{\frac{1}{2}}$; $v_2 = -2r_2 \times (2\pi Sn)^{\frac{1}{2}}$; $v_3 = c(2\pi Sn)^{\frac{1}{2}}$; $\omega = 2(2\pi Sn)^{\frac{1}{2}}$; and $\nu = (2Sn)^{\frac{1}{2}}/\pi$. These forms together are designated (77).

CHECKING THE ANALYTICAL RESULTS

Accuracy of the derivations was checked by dimensional inspection, interchecking among the various results, substitution in the energy equation, and experimentally as described below.

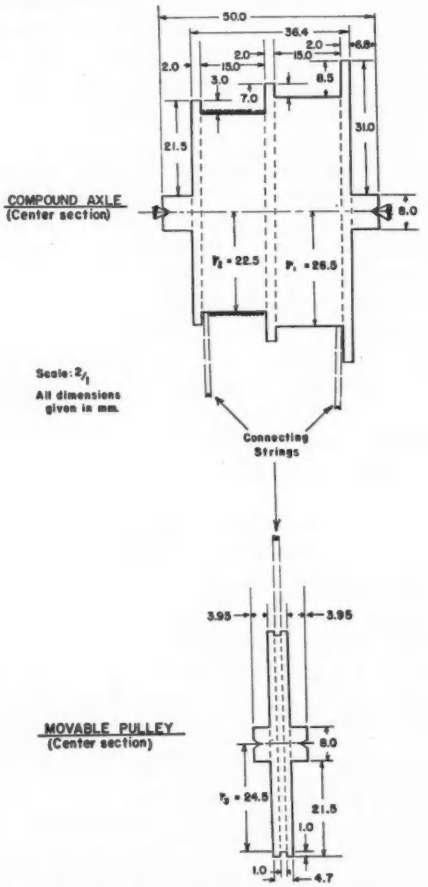

FIG. 3. Vertical axial section of fixed compound axle and movable pulley, to scale. The string is shown unwound from larger drum of axle, so that pulley is at its lowest level. Although pulley is shown here with its plane perpendicular to plane of paper, actually in this lowest position its plane has turned about a vertical axis through some 30° from its initial approximate coplanarity with "plane of axle," here perpendicular to plane of paper.

APPARATUS

The compound axle was lathed from a single piece of solid aluminum, and similarly the pulley. The vertical section, Fig. 3, is to scale. The string, over 9 ft long, had its ends fixed to the drums by drilling a hole from the periphery of each drum next the flange obliquely through the axle, to emerge through the outside flange-plane of the larger drum, at the right (Fig. 3), so that the two emergence openings are diametrically opposite on this face. Each opening is approached by the wound-up cord in a gradually deepening gouge in the flange, the fishline string keeping its helical path until it disappears into the flange. The end segments of the string were threaded through the holes from the drum peripheries and knotted. In effect, the cord was attached to each drum on its periphery. Each successive turn has close contact with the preceding and the following turn. Strictly, the 1-mm diameter of the cord should be added to diameters of drums and pulley, for calculating values in checking the equations, but the error is small. In fact it affects r_1+r_2 but not r_1-r_2, so that neither of the functions Q and S is affected.

Figure 4 gives comparison photographs of the apparatus, made at four different levels of the pulley during its descent. The agreement in phase is evident.

NUMERICAL VALUES OF PARAMETERS

Referring to Fig. 3, $r_1 = 2.65$, $r_2 = 2.25$ cm; $b = 4.9$, $c = 0.4$ cm; $m = 65$ gm; $g = 980$ cm/sec^2. Calculated: $I_a = 705$, $I_p = 69$ g-cm^2, treating the axle as an assemblage of solid cylinders. From these values, $I_e = m(r_1 - r_2)^2 = 10.8$, while $Q = 3107$, $S = 8.2008$. These precise values for Q, S assume similar precision in measuring the independent parameters. Further abbreviating functions are: $A = I_a b + 2I_p r_2 = 3765.0$, $B = I_a b + 2I_p r_1 = 3820.2$ absolute cgsu.

Minimum vertical distance center to center of pulley and axle, 9.7 cm; maximum distance after 16 turns, $9.7 + 20.1 = 29.8$ cm. Because of the 2-mm thickness of the middle flange, Fig. 3, the plane of the pulley at its upper limiting position lacks by calculation and measurement about 3° of being parallel with the "plane of the axle," or plane of the analysis. At the lower limiting

position reached after the maximum 16 turns, the pulley plane is inclined about 33°, since it veers steadily during the descent. However, even here, as calculation shows, the free lengths of the string do not depart from the vertical by as much as 1°, which accounts for the pulley's not being perceptibly out of phase with the axle even after reaching the lowest level, after 16 turns of axle and pulley, Fig. 4. In Fig. 3, the vertical section of the pulley perpendicular to its plane is shown as though it remained parallel with the axle throughout the descent.

Applying some of the derived equations, which assume no friction, we have by Eqs. (32) and (39), $a_3/g=0.003347$, or the linear acceleration of the pulley is only about 1/300 that of free fall. Also by Eq. (20), the sum of the string tensions f_1+f_2 is 99.665 percent of the pulley's weight. This value is consistent with the pulley's low linear acceleration. Whereas its weight is 65.000 g, the total upward pull of the string on the pulley is 64.782 g. By Eq. (25), ratio $f_2/f_1=B/A=1.0147$, so that the tension f_2 is only about 1.5 percent greater than f_1, while by Eq. (48), the ratio of torques on the axle is $\tau_1/\tau_2=1.1607$, τ_1 being 16 percent greater than τ_2, in spite of the force f_1 on that side, the left, being less than f_2 on the right. Tension $f_1=32.155$, while $f_2=32.627$ g. (For further data see section "Numerical Table" and Table I near end of paper.)

EXPERIMENTAL CHECKING

The calculated h_3 for 15 turns was 18.8 cm, Eq. (65), which checked with direct measurement. With a stop watch, not a precise method considering shortness of the measured interval, the pulley's descent through 15 turns was timed. However, several successive measurements agreed well and the measured time was taken as 4.5 sec. The calculated time, Eq. (68), was 3.38 sec. An error in this direction was expected, because friction slows the descent.

Here two questions may be raised: (1) If the apparatus had no friction how much would the total moment of inertia $I=I_a+I_p$ have to be increased to give the observed time of descent of 4.5 sec, the system remaining otherwise the same? By Eq. (68), the new value of S would be $2h_3/(ct^2)=2\times18.849/(0.4\times20.25)=4.6541$. Then by Eq. (32), $Q=cmg/S=0.4\times65\times980/4.6541=5474.7=4I+mc^2$, by Eq. (29); therefore, the required new value would be $I=1366$. The actual value calculated from measurements was 774. There would have to be an increase of 76.5 percent in the combined moment of inertia to slow the descent as much as was actually caused by friction.

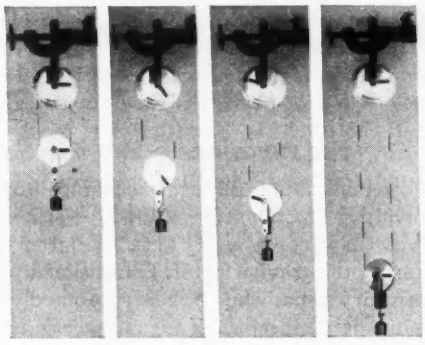

FIG. 4. Photographs of apparatus made at four different levels of pulley during its descent under gravity. Radial markers show that pulley and compound axle remain in phase. Also, visibility of the light-colored string is increased by intermittent inking. In (d), with pulley at lowest level, inclination of its plane from the initial orientation in (a) is apparent. This obliquity causes in the photograph an apparently much greater departure from horizontal parallelism of the two radial markers than actually exists.

(2) What is the equivalent force of friction? That is, assuming no friction what external force F applied vertically upward on the pulley would slow the descent down from the ideal calculated value of 3.38 sec to the observed time, 4.50 sec? Since distribution of friction in the system is unknown, f_1 and f_2 cannot be calculated and we resort to energy relations. In the descent of the pulley through $h_3=18.8$ cm, for 15 turns from rest, loss of PE is $mgh_3=65\times980\times18.8=1{,}200{,}000$ ergs. The actual KE of the whole system is $I\omega^2/2+mv_3^2/2$. By Eq. (16), $\omega=(v_1-v_3)/r_3$, and since $v_1=\omega r_1$, we have $v_3=\omega c/2$. Then $\text{KE}=(I+mc^2/4)\omega^2/2$, where the second term is small. To find $\omega=\alpha t=a_1t/r_1$, we have from Eqs. (1) and (10), $a_1=2r_1a_3/c$, where $a_3=2h_3/t^2$, t being the observed time. There results $a_3=1.85$, $\omega=2a_3t/c=2\times1.85\times4.5/0.4=43.9$. Then $\text{KE}=(774+65\times.16/4)\times1927/2=748{,}000$ ergs, which is 62.3 percent of the loss

of PE. The difference 452,000 ergs = 0.011 cal, or 37.7 percent, is friction loss. The equivalent friction force is then $F = 452{,}000/18.8 = 24{,}000$ dynes = 24.6 g, 38 percent of the pulley weight.

THE FUNCTIONS Q AND S

Let us attribute physical meanings to the abbreviating functions: $Q = 4(I_a + I_p) + m(r_1 - r_2)^2$ (29); and $S = (r_1 - r_2)mg/Q$ (32), the latter only containing the gravity factor. Dimensionally Q is moment of inertia, while S, being torque divided by moment of inertia, is angular acceleration. Physical interpretation for S is left to the reader. As we shall see, Q is a measure of the total inertial reaction of the system, while S is affected by the actuating force mg.

Although S is proportional to g, it is not proportional to m, or mg, because m appears also in one term of function Q. The tensions in the string, f_1, f_2, and all accelerations, velocities, and displacements, both linear and angular, are proportional to S. Thus all these are proportional to g but not to m, although their values are affected by it.

This absence of direct proportionality of cord tensions and accelerations to the driving force is met in the analogous simpler system of two masses connected by a string passing over a pulley, the larger mass resting on a smooth table top. The whole system accelerates according to the second law, $a = mg/(M+m)$. If M is very large compared with m we have a good comparison with the present case, where $m(r_1 - r_2)^2$ in function Q is only about $\frac{1}{3}$ percent of Q.

Suppose further $a = a_3$ ($= 3.3$ cm/sec^2), or $mg/(M+m) = (r_1 - r_2)S$, by Eq. (39). Then from Eqs. (29) and (32), $M = Q/(r_1 - r_2)^2 - m = 3107/0.16 - 65 = 19\ 354$ g, or about 42.5 lb being accelerated by a weight of 65 g. That is, of the whole inertial reaction, the larger part due to the relatively large moments of inertia of axle and pulley is equivalent in the analogous system to $M = 42.5$ lb on the table top, while a weight of 65 g accelerates both masses. Ratio $m/(M+m)$ equals ratio $mc^2/Q = a_3/g$ in the present system.

To discuss Q from another point of view we note that $Q/4 = (I_a + I_p) + m[(r_1 - r_2)/2]^2 = I + I_e$, where $I_e = m[(r_1 - r_2)/2]^2 = mr_e^2$. Then $Q/4$ is the sum, total moment of inertia I plus that of an *equivalent particle* of mass m at distance $r_e = (r_1 - r_2)/2$ from an axis. Suppose this particle has an angular acceleration α equal to that of the axle and pulley in common. Then $\alpha = a_e/r_e = 2S$, by Eq. (45), so that $a_e = (r_1 - r_2)S = cS$, where a_e is the tangential acceleration of the equivalent particle. Then by Eq. (39), $a_e = a_3$, which also is obviously the tangential acceleration of any point on the axle located at distance r_e from rotation axis. Reversing this reasoning, from Eq. (39) $a_3/g = m[(r_1 - r_2)/2]^2/(Q/4) = I_e/(I + I_e)$. Thus $Q/4 = I + I_e$, the relation started with above.

Function Q is *a measure of the total inertial reaction in the system*, in terms exclusively of three moments of inertia—those of axle and pulley together with that of the equivalent particle.

EQUIVALENT AXLE

Referring again to the equivalent particle, since $a_e = a_3$ is the tangential acceleration of any point located on the axle at distance $r_e = (r_1 - r_2)/2$ from its axis, imagine next a third drum attached coaxially to the compound axle, with radius r_e but negligible mass, so that it has no effect on the motions in the system. Let a weightless cord be wrapped around it, with one end fastened to the drum, and a small weight, negligible but just enough to keep the string taut, attached to the free end. If initially this small weight is hung at the same level as the pulley axis, but not touching it, then after release of the system the weight, accelerating at rate $a_e = a_3$, will remain on the same level as the pulley. For the present apparatus, the diameter of this imaginary drum is $c = r_1 - r_2 = 4.0$ mm, which contrasts with diameters of the actual drums, 26.5 and 22.5 mm.

Now suppose this "equivalent axle" is an actual drum replacing the whole system. Let the mass hanging from the string be m, the same as that of the replaced pulley. We ask, what moment of inertia I_e would be necessary for $a_e = a_3$. We have $\tau_e = mg \cdot r_e = mg(r_1 - r_2)/2 = I_e\alpha = I_e \cdot 2S$ by Eq. (45) so that $I_e = mgr_e/(2S) = Q/4$, the total inertial reaction of the replaced system, in terms exclusively of moment of inertia. Thus, wherever we meet the factor c in the equations, we can be reminded of this equivalent axle, the torque on which is $\tau_e = QS/2 = cmg/2$.

TWO ROLES OF THE PULLEY MASS

In the closed differential pulley when operating, the mass m of the pulley unit plays two separate roles: (1) in the force factor mg its role is merely to supply a material object the weight of which accelerates the system; (2) in the m-term, $m(r_1-r_2)^2$, present in the function Q—this m-term being 4 times the moment of inertia of the equivalent particle—its role is to contribute to the total inertial reaction of the system.

A similar situation exists in the analogous case where $a=mg/(M+m)$. Here the role of m in the numerator, where it determines the applied force, is different from that in the denominator, where it presents inertial reaction. Thus, in general, any mass has the two roles mentioned, whether or not it is part of a complicated system. The present apparatus could be laid on a smooth table top, and a force F_a applied to the pulley along the line of centers of pulley and axle that would be entirely independent of its mass, thus eliminating the first role of m while retaining the second. This means substituting F_a for mg in Eq. (20). If from this point, further derivation proceeds on this basis, the same expressions are obtained throughout, except that F_a now appears in place of mg. The inertial term $m(r_1-r_2)^2$ for the linear motion of the pulley remains.

ADAPTATION TO STUDENT LABORATORY

By suitable selection from the working equations, the closed differential pulley is available for student laboratory. Procedure can be simplified for lower division classes, or made more elaborate for an advanced course in mechanics. For the latter, derivations can be required.

The vertical distance of descent of pulley h_3 is measurable quite precisely. If a stop watch is used a longer path of descent is desirable. This can be supplied with apparatus of the present dimensions by using finer fishline or strong thread, giving a correspondingly larger number of turns on the axle, or by increasing somewhat the widths of the drums, while retaining the 1-mm cord.

To prevent jerking when the pulley reaches its lowest position, where the cord starts abruptly to wind again on the large drum, a simple brake can be quickly applied to the axle just before the limiting instant, and also a small platform can be placed in the pulley path.

By weights one can vary the mass of the pulley unit to control the descent time over a chosen distance. Thus Eq. (68), where S depends on m, can be the working formula. If h_3 is to be treated as a variable parameter, the relative values of the independent parameters can be changed from the particular ones here, in order to give a wider spread in values of t.

The experiment can include measuring parametric quantities r_1, r_2, I_a, I_p, and m. Moment of inertia I_a can be calculated, as was done here, on the basis of the compound axle's being an assemblage of coaxial solid cylinders, while if the pulley is a solid cylinder, I_p is calculated similarly. Thus, the student himself finds the necessary numerical parameters. For advanced laboratory the experiment has rather extensive possibilities.

SUGGESTED EXERCISES

The descending pulley offers practice with vectors, if attention is fixed on a point P at the pulley rim. Particularly choose the point which at $t=\phi=0$ was at the right end of a horizontal diameter. This point revolves counterclockwise according to Fig. 1, about the center of the pulley as it descends, its position on the circle at any instant being given by the phase angle ϕ. Let the origin of Cartesian coordinates be at the position of the pulley center when at the highest level. The origin is thus stationary with respect to the ground. If the vector is drawn full scale its tail can be at the origin while the head remains with the moving point P. The motion of point P, particularly with respect to the ground, is somewhat complicated.

Exercise 1. In time t what is the instantaneous displacement of point P from the origin? The displacement vector changes length while oscillating through the y axis. The x component is $r_3\cos\phi$, while $y=h_3-r_3\sin\phi$, whence by Eq. (63), $y=(c\phi-b\sin\phi)/2$, downward direction positive. From these the resulting vector has length $R=(b^2-2bc\phi\sin\phi+c^2\phi^2)^{\frac{1}{2}}/2$. The direction angle ψ with the y-axis is given by $\tan\psi=x/y$.

Exercise 2. What is the instantaneous velocity of point P with respect to the ground, that is with respect to the origin? It is the vector sum of the tangential velocity v_T of P with respect to the pulley center and the velocity v_3 of the pulley

TABLE I. Calculated values of variables dependent on time. (cgs absolute units)

n	ϕ	t	ω	v	h_1	$-h_2$	h_3	v_1	$-v_2$	v_3
0	0	0	0	0	0	0	0	0	0	0
1	6.25	0.88	14.4	2.29	16.7	14.1	1.26	38.0	32.3	2.87
2	12.6	1.24	20.3	3.23	33.3	28.3	2.51	54.3	45.7	4.06
3	18.8	1.52	24.9	3.96	50.0	42.4	3.77	65.9	55.9	4.97
4	25.1	1.75	28.7	4.57	66.6	56.5	5.03	76.1	64.6	5.74
5	31.4	1.96	32.1	5.11	83.3	70.7	6.28	85.1	72.2	6.42
6	37.7	2.14	35.2	5.60	99.9	84.8	7.54	93.2	79.1	7.03
7	44.0	2.32	38.0	6.05	117	99.0	8.80	101	85.5	7.60
8	50.3	2.48	40.6	6.46	133	113	10.1	108	91.4	8.12
9	56.5	2.63	43.1	6.86	150	127	11.3	114	96.9	8.61
10	62.8	2.77	45.1	7.23	167	141	12.6	120	102	9.08
11	69.1	2.90	47.6	7.58	183	156	13.8	126	107	9.52
12	75.4	3.03	49.7	7.92	200	170	15.1	132	112	9.95
13	81.7	3.16	51.8	8.24	216	184	16.3	137	116	10.4
14	88.0	3.28	53.7	8.55	233	198	17.6	142	121	10.7
15	94.2	3.39	55.6	8.85	250	212	18.8	147	125	11.1
16	100.5	3.50	57.4	9.14	266	226	20.1	152	129	11.5

center with respect to ground. Basically, this is the problem of a boat on a river that moves with respect to the bank. Here, however, both component vectors are variable. By Eqs. (72) and (73), $v_T = 2r_3(S\phi)^{\frac{1}{2}} = b(S\phi)^{\frac{1}{2}}$, and by Eq. (71), $v_3 = c(S\phi)^{\frac{1}{2}}$.

Exercise 3. What is the instantaneous linear acceleration of point P with respect to the ground? Here three vectors are added: (1) radial acceleration, toward the pulley center, $a_c = r_3\omega^2 = 2bS\phi$, by Eqs. (72) and (73); (2) tangential acceleration, $a_T = r_3\alpha = bS$ by Eq. (45); and (3) acceleration of the pulley center itself, $a_3 = cS$ by Eq. (39). This resultant vector presents a challenge, requiring both length and direction as functions of ϕ. The length of the first of the three components depends on time, that is on ϕ, while the lengths of the others do not. However, directions of both the first two depend on ϕ, while that of the third does not, since that component is constant.

Exercise 4. The actual path traversed by point P with respect to the ground is followed by the head of the displacement vector, if it is drawn full scale and treated as above. What is the equation of this path? Procedure calls for elimination of ϕ between expressions for components x, y, as given under Exercise 1.

Exercise 5. What is the kinetic energy of twist of the pulley about a vertical axis, for a given h_3? This is obviously small, but how small?

NUMERICAL TABLE

Assuming no friction in the present apparatus, Table I gives numerical values dependent on time, calculated in terms of total rotations n from the start. Formulas will be found in the group designated Eq. (77) above. The time relation is that of Eq. (75). (See "Numerical Values of Parameters" above.)

Values of dependent parametric quantities are: (a) string tensions, by Eqs. (33) and (34): $f_1 = 31{,}507$, $f_2 = 31{,}969$ dynes, (note that sum $f_1 + f_2 = 64.771$ g is only slightly less (0.35 percent) than weight of pulley, 65.000 g, while $f_2 - f_1$, Eq. (35), is only about 0.73 percent of mg); (b) torques, by Eqs. (40), (41), (46), and (47): on axle, $\tau_1 = 83{,}493$, $\tau_2 = -71{,}931$, and $\tau_a = \tau_1 + \tau_2 = 11{,}563$, which checks with $2I_aS$; on pulley, $\tau_p = 2I_pS = 1160.0$, checking with $(f_2 - f_1)r_3$, Eq. (41); and finally, (c) accelerations, by Eqs. (37), (38), (39), and (45): $a_1 = 43.5$, $a_2 = -36.9$, $a_3 = 3.28$ cm/sec^2; while $\alpha = 16.40$ rad/sec^2.

POINTS OF INTEREST

(1) Of primary importance to teaching is the fact that the apparatus is mechanically simple enough for the student to follow the working out, in a rather complicated way, of Newton's second law, operating here in simultaneous, related linear and angular motions. A considerably larger number of magnitudes is met than in the Atwood's machine.

(2) Mathematical relations include three different-valued linear motions, having to do with accelerations, velocities, and displacements, and also corresponding to these the angular quantities of two different rotary units, one fixed, the other having translation. Although many equations appear during the derivations, the working formulas are not too numerous and are quite simply expressed.

(3) The abbreviating functions Q and S have physical meaning as well as mathematical utility.

(4) Interpretation can be made in terms of an "equivalent axle," the diameter of which equals the difference in radii of the drums of the compound axle.

(5) The distance the pulley descends in one rotation is equal to the difference of radii of axle drums multiplied by π. In the present apparatus the pulley rotates nearly four times to descend a distance equal to its own diameter.

(6) Although the tangential *force* on the small drum of the axle, which is the tension in the right string, Fig. 1, is *greater* than that on the large drum, which is the tension in the left string, the

torque on the small drum is *less* than that on the large one. The ratio of the two torques is equal to the ratio of the two tensions multiplied by the ratio of the radii. This product here is $-0.986 \times 1.18 = -1.16$, the tensions differing from each other by only 1.5 percent, while the radii differ by 18 percent, causing the torques to differ by 15 percent, oppositely from the tensions.

(7) Two visible characteristics of the mechanical action: (a) the cord winds up on one side at the same time it unwinds on the other, giving somewhat the impression of a "self-acting" mechanism, but an "externally applied force" is necessary; (b) the descending pulley remains in phase with the fixed rotating axle, analysis showing that pulley and axle have the same angular acceleration and velocity at every instant.

(8) Along with the Atwood's machine and the inclined plane, this system offers a mechanical means of reducing acceleration of gravity, so that a/g can be varied. Here the acceleration of the pulley is about $\frac{1}{3}$ percent of g. A smooth inclined plane giving this acceleration to a mass sliding down it, would need to be tilted from the horizontal only 11.5 mins of arc. Correspondingly in the Atwood's machine, neglecting friction and moment of inertia, the mass of the rider would be 0.66 percent of the mass of one of the equal weights.

CONCLUSION

In general physics, "simple machines" and mechanical advantage are treated as elementary problems in equilibrium, any accelerations being usually omitted. These systems, especially for the college student, have taken on something of a "cut-and-dried" nature. Their basic importance, however, needs continued emphasizing. This paper proposes that the classical systems of pulleys be dealt with from a nonequilibrium point of view.

A particular pulley system is proposed and described. From the nonequilibrium standpoint it offers broad opportunities for instructional use in lecture, quiz, classroom, and laboratory. The paper analyzes the system mathematically while emphasizing the physics. The resulting working equations are summarized. The derivations, in a particular sequence, keep the beginning student in mind.

The student, finding himself here in what may seem at first a somewhat baffling experimental situation, by continued "looking beneath the surface" gradually recognizes an increasing number of related mechanical facts, assuming he has had no information about the system beforehand. At the same time he observes in this example nature's ability to "hang together," avoiding entanglements and maintaining a "balanced" condition amid complexity, through an impressively consistent use of simply stated principles, especially that of Newton's second law.

Provocative questions, problems and exercises have been included for use by teachers. The plotting of graphs, like $I_a = f(m)$, the numerous other quantities including time remaining the same, would add interest. Obviously, teaching possibilities have not been exhausted in the present paper.

What spurred the present study was the request made by the Secretary, Southern California Section of AAPT, for program papers preferably in the fields of alternating currents, Newton's second law and rotation. The last named subject appealed particularly to this writer.

Finally, thanks are due to Mr. Lewis H. Humason for his assistance in the mechanical design, and to the Physics Shop for construction of the apparatus.

We, like the eagles, were born to be free. Yet we are obliged, in order to live at all, to make a cage of laws for ourselves and to stand on the perch. We are born as wasteful and unremorseful as tigers; we are obliged to be thrifty, or starve, or freeze. We are born to wander, and cursed to stay and dig.
—WILLIAM BOLITHO.

The Uniform Model of the Nucleus*†

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The uniform model of the nucleus, proposed by Wigner in 1937, assumes that the motion of the nucleons within the nucleus are so complicated, and the interactions between them are so strong, that the details of the motion are of no great importance. This model, therefore, differs diametrically from the shell model of the nucleus, in which each nucleon is assumed to move independently of all others to a first approximation, the various nucleons filling up a regular sequence of possible orbits (shells). In this paper we re-examine the evidence for the uniform model to see whether, in the light of data accumulated since the time the model was proposed, the agreement between the predictions of the uniform model and experimental binding energies is sufficiently good to serve as an argument against the shell model. This is found not to be the case. The uniform model gives qualitative agreement with experiment, but the detailed quantitative agreement is only moderately good. This rather indifferent record must be compared with the excellent evidence for nuclear shell structure in the same region of mass numbers. Nevertheless, we feel that the uniform model should not be abandoned altogether, but rather should be reinterpreted to apply to the "characteristic state" postulated by Hurwitz and Bethe, rather than to the ground state. No test of this hypothesis is possible until more is known experimentally about the level spectra of nuclei. In this connection a modified semiempirical form is suggested for the density of nuclear levels as a function of excitation energy.

1. INTRODUCTION

THE uniform model of the nucleus, proposed by Wigner,[1] assumes that the motion of the nucleons within the nucleus is so complicated, and the interactions between them are so strong, that the details of the motion are of no great importance. The interactions between nucleons in the nucleus are assumed to be given by exchange forces of the Majorana (space exchange) type, with perhaps a small admixture of ordinary (Wigner) forces. Spin-dependent forces are assumed to be small enough so that they can be neglected in a first approximation. The contribution of each pair of nucleons to the total potential energy then depends only upon the type of the pair, i.e., upon whether the wave function is symmetric or antisymmetric under the exchange of the space coordinates of the two nucleons, and upon the nuclear radius.

Wigner[1] and Barkas[2] have shown that the theory based upon these assumptions gives excellent agreement with experimental data on binding energies, especially for the differences between binding energies of neighboring isobaric nuclei. Margenau and Wigner[3] showed that the observed magnetic moments of nuclei could also be understood on the basis of the uniform model. Wigner[4] applied the theory to the calculation of the beta-decay constants (*ft* values) of "allowed and favored" beta-transitions in light nuclei, again obtaining excellent agreement.

Thus, some ten years ago, the uniform model was considered very successful. However, various developments since that time have made a reexamination of the model desirable. First, Wigner[1] already called attention to the fact that the model underestimates the absolute amounts of the binding energies, even though it gives a good approximation to the difference between binding energies of neighboring isobars. Second, the discovery of the quadrupole moment of the deuteron[5] and its interpretation in terms of a non-

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‡ Part of this work was done while both authors were at Massachusetts Institute of Technology. Additional research to complete the paper was performed while the second author was at the University of Illinois.
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central, spin-dependent tensor force[6] made the assumption of spin-independent forces somewhat doubtful. Recent calculations of Feingold[7] on light nuclei indicate that the tensor forces make important contributions to the total binding energy. Third, the high energy neutron-proton and proton-proton scattering data obtained at Berkeley show that the basic law of force between two nucleons is very much more complicated than had been believed. The scattering data seem to indicate that the nucleon-nucleon force does not have enough exchange character to lead to saturation in heavy nuclei.[8] While these lines of evidence tend to weaken the basic assumptions of Wigner about the force law, the recent evidence for shell structure in nuclear ground states and in low-lying excited states[9] weakens the "uniformity" assumption quite badly. The shell structure evidence indicates that the nucleons move, to a first approximation, independently of each other in well-defined orbits. This is the exact opposite of the "uniform" assumption.

Thus, it appears quite surprising that the uniform model should be so extremely successful in predicting nuclear binding energies. A re-examination of the evidence by Frisch,[10] using more recent experimental data, did indeed indicate somewhat poorer agreement. The present investigation was undertaken to see to what extent the agreement of the uniform model with experiment can be considered proof of the "uniformity" assumption, and therefore as an argument against the usual explanation of nuclear shell structure in terms of an independent particle model of the nuclear motion. Since there are theoretical reasons for expecting the uniform model to break down for very light nuclei on the one hand (the uniformity assumption surely fails there) and for heavy nuclei on the other hand (Coulomb effects become very important), we have restricted ourselves to nuclei with mass numbers between 16 and 65. The precise range chosen is somewhat arbitrary, but a line had to be drawn somewhere in order to avoid the temptation to include "just one more good case." We have compared the observed binding energy differences between neighboring isobars and the total binding energies of nuclei within that range with the predictions of the uniform model.

The result can be summed up as follows: The uniform model gives qualitative agreement with experiment, but a more quantitative comparison shows that the detailed agreement is only moderately good. In particular, the effects of magic numbers appear very clearly, and even for nonmagic nuclei the uniform model has a tendency to underestimate the amount by which even-even nuclei are favored over odd-odd nuclei. The old disagreement with the total binding energies remains. This rather indifferent record of the uniform model should be compared with the excellent evidence for nuclear shell structure in the same region of mass numbers. In our opinion, the comparison speaks against the uniformity assumption for the ground states of nuclei. Nevertheless, we feel that the uniform model should not be abandoned altogether, but rather should be reinterpreted to apply to the "characteristic level" postulated by Hurwitz and Bethe[11] rather than to the ground state. This suggestion is discussed in Sec. 5 of this paper. No definite conclusions can be reached until more is known experimentally about excited states of nuclei.

2. THE BINDING ENERGY FORMULA OF THE UNIFORM MODEL

In this section we compile the relevant formulas for easier reference. The total energy of a nucleus consists of three parts: the nuclear potential energy V, kinetic energy T, and electrostatic (Coulomb) potential energy C. The nucleus is specified by stating its mass number A, and its neutron excess T_z defined by

$$T_z=\tfrac{1}{2}(N-Z), \tag{2.1}$$

where N is the number of neutrons and Z is the number of protons. The potential energy for a

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[11] H. Hurwitz and H. A. Bethe, Phys. Rev. **81**, 898 (1951).

pair of nucleons is assumed to be

$$V_{12}=V_W(r_{12})+V_M(r_{12})P_{12}{}^M, \qquad (2.2)$$

where V_W is the potential of the Wigner force, V_M gives the distance dependence of the Majorana exchange force, and P^M is the Majorana operator which exchanges the positions in space of the two nucleons in question, without affecting the spin coordinates.

We distinguish between "symmetric" and "antisymmetric" pairs of nucleons (under the operation of space-exchange). We denote by $V_W{}^s$ the average value of $V_W(r_{ij})$ for symmetric pairs, and by $V_W{}^a$ the corresponding average for antisymmetric pairs. A similar notation is used for averages of the Majorana potential. Let n_+ be the number of symmetric pairs, n_- the number of antisymmetric pairs. Then the nuclear potential energy is given by

$$V=n_+V_W{}^s+n_-V_W{}^a+n_+V_M{}^s-n_-V_M{}^a. \qquad (2.3)$$

This expression can be rewritten as

$$V=-L(n_+-n_-)-L'(n_++n_-), \qquad (2.4)$$

where

$$L=-\tfrac{1}{2}(V_W{}^s-V_W{}^a)-\tfrac{1}{2}(V_M{}^s+V_M{}^a)=L_1/A, \qquad (2.5)$$

$$L'=-\tfrac{1}{2}(V_W{}^s+V_W{}^a)-\tfrac{1}{2}(V_M{}^s-V_M{}^a)=L_1'/A. \qquad (2.6)$$

The uniform model of Wigner assumes that the quantities L and L' are smooth functions of the mass number A, independent of special properties of individual nuclei. Furthermore, both L and L' can be estimated to be inversely proportional to A, so that the quantities L_1 and L_1' defined by Eqs. (2.5) and (2.6), respectively, are taken to be constants. While these constants may be estimated from our information about forces between isolated nuclear particles, they are usually taken to be adjustable parameters in this theory. The signs in Eqs. (2.5) and (2.6) have been chosen in such a way that L and L' are expected to be positive.

The form of Eq. (2.4) is useful since the second term is proportional to the total number of pairs,

$$n_++n_-=\tfrac{1}{2}A(A-1), \qquad (2.7)$$

and may hence be ignored in comparisons of binding energies of isobaric nuclei. The crucial quantity in that connection is the difference n_+-n_- between the numbers of symmetric and antisymmetric pairs.[12] This quantity can be derived by group-theoretical methods. For any one nucleus it can assume many different values, the spectrum of possible values depending upon A and T_z. According to Eq. (2.4) we are interested in the highest eigenvalue of n_+-n_- consistent with a given mass number and neutron excess; for this gives the minimum potential energy V and, hence, is appropriate for the ground state of the nucleus. Physically this is reasonable since the Majorana force is attractive for symmetric pairs, repulsive for antisymmetric pairs, so that large values of n_+-n_- give large binding energies. This highest eigenvalue of n_+-n_-[13] is given by

$$n_+-n_-=-(A^2/8)+2A+(5/2)-\xi, \qquad (2.8)$$

where ξ depends upon the type (even-even, odd-odd, even-odd) of the nucleus in question:

$$\xi=\tfrac{1}{2}T_z{}^2+2|T_z|+(13/4) \quad \text{odd } A, \qquad (2.9a)$$

$$\xi=\tfrac{1}{2}T_z{}^2+2|T_z|+(5/2) \quad \text{even } A, \text{ even } Z, \qquad (2.9b)$$

$$\xi=\tfrac{1}{2}T_z{}^2+2|T_z|+4 \quad \text{even } A, \text{ odd } Z, \text{ except } Z=N, \qquad (2.9c)$$

$$\xi=5 \quad \text{even } A, \text{ odd } Z, \text{ with } N=Z. \qquad (2.9d)$$

We observe that ξ is lowest for $|T_z|$ close to zero. This is in agreement with the empirical rule that nuclei prefer to have equal numbers of neutrons of protons, except for the Coulomb effect which favors a neutron excess. In a similar way, the following three empirical rules are also direct consequences of Eqs. (2.8) and (2.9): (1) The binding energy of odd-A isobaric nuclei is a smooth function of the neutron excess T_z; (2) even-even nuclei are more strongly bound than odd-odd nuclei; (3) odd-odd nuclei with $N=Z$ form an exception to rule (2), several of them being beta-stable. Thus, Wigner was able to show that the assumption of spin- and charge-independent forces of predominantly Majorana exchange character together with a uniform

[12] This quantity is called Ξ in reference 1. The ξ of Eqs. (2.8) and (2.9) is Wigner's Ξ'.

[13] Under Wigner's assumptions about the nuclear forces (charge- and spin-independent), and with the further assumption that the Coulomb repulsions between the protons are negligible in a first approximation, the operator n_+-n_- commutes with the Hamiltonian H of the nucleus.

model for the nuclear motion provides a natural explanation not only of saturation, but also of the empirical rules for nuclear stability enumerated above.

The relations (2.5)–(2.9) contain the estimate of the uniform model for the nuclear potential energy. The kinetic energy T is estimated by assuming that the nucleons form a Fermi gas confined to a sphere of the nuclear radius

$$R=r_0A^{\frac{1}{3}}=1.47A^{\frac{1}{3}}\times10^{-13}\text{ cm}. \quad (2.10)$$

The kinetic energy of the Fermi gas also depends upon the neutron excess to some extent, since the height of the Fermi distribution is increased by an increased number of identical particles. For $|T_z|\ll A$, we get the approximate relation

$$T=T_0A+(T_1/A)\eta+\cdots, \quad (2.11)$$

where T_0, T_1, and η are defined as follows (M = nucleon mass):

$$T_0=(3/40)(9\pi)^{\frac{1}{3}}(\hbar^2/Mr_0^2)\cong13.5\text{ Mev}, \quad (2.12)$$

$$T_1=(40/9)T_0\cong60\text{ Mev}, \quad (2.13)$$

$$\eta=\tfrac{1}{2}T_z^2+\tfrac{1}{4} \quad \text{odd } A, \quad (2.14a)$$

$$\eta=\tfrac{1}{2}T_z^2 \quad \text{even } A, \text{ even } Z, \quad (2.14b)$$

$$\eta=\tfrac{1}{2}T_z^2+\tfrac{1}{2} \quad \text{even } A, \text{ odd } Z, \text{ except } Z=N, \quad (2.14c)$$

$$\eta=\tfrac{1}{2} \quad \text{even } A, \text{ odd } Z, Z=N. \quad (2.14d)$$

The numerical estimates in Eqs. (2.10), (2.12), and (2.13) are derived from the study of the Coulomb energy difference between mirror nuclei, which depends upon the nuclear radius. The value 1.47×10^{-13} cm for r_0 is also in reasonable agreement with data from nuclear reactions. The terms neglected in Eq. (2.11) are of order $(T_z/A)^2$ or smaller. They are negligible compared to the intrinsic error of the Fermi gas approximation. Indeed, the estimate of the kinetic energy is probably one of the weaker points of the theory. One must remember that about half the potential energy of attraction in the nucleus is compensated by the kinetic energy, so that an error in the estimate of the kinetic energy is very serious indeed for the binding energy.

Since T_1 in Eq. (2.11) is positive, nuclear stability is enhanced by a low value of η. Equations (2.14) show that η has properties which are in many ways similar to ξ, Eq. (2.9).

Finally, we need an expression for the Coulomb energy of the nucleus. In accordance with the "uniform" assumption, the Coulomb energy is estimated as that of a uniform distribution of Z protons over a sphere of the nuclear radius R, Eq. (2.10), i.e.,

$$C=\tfrac{1}{2}Z(Z-1)(6e^2/5R)=4u_cZ(Z-1)A^{-\frac{1}{3}}. \quad (2.15)$$

Collecting the expressions (2.4), (2.7), (2.8), (2.11), and (2.15), we can write the total energy $E(A, T_z)$ of a nucleus as the sum of two terms, the first of which depends only upon the mass number, while the second contains the stability relations between isobaric nuclei:

$$E(A, T_z)=E_0(A)+E_1(A, T_z), \quad (2.16)$$

where $E_0(A)$ can be chosen to be

$$E_0(A)=L_1(A/8-2-5/2A)-\tfrac{1}{2}L_1'(A-1)+T_0A+u_cA^{5/3}(1-2/A). \quad (2.17)$$

With this choice of $E_0(A)$, $E_1(A, T_z)$ becomes

$$E_1(A, T_z)=(L_1\xi+T_1\eta)/A-4u_cA^{-\frac{1}{3}}T_z(A-1-T_z). \quad (2.18)$$

In this way of writing the nuclear energy, there are five adjustable parameters: L_1, L_1', T_0, T_1, and u_c. Wigner[1] took the following approach: u_c was taken from the Coulomb energy differences between mirror nuclei (ξ and η in Eq. (2.18) are the same for two mirror nuclei, T_z and $-T_z$), and T_0, T_1 were taken from the Fermi gas approximation for the kinetic energy. This left only two parameters, L_1 and L_1', only one of which (namely, L_1) was of importance for the determination of the stability relations between isobars. We shall allow ourselves more latitude in fitting the data: we also take u_c from the mirror nuclei, but we use the Fermi gas approximation for the kinetic energy only qualitatively, i.e., in so far as it determines the *form* of the dependence of kinetic energy upon A and T_z. The quantities T_0 and T_1 will be treated as adjustable parameters. According to Eq. (2.18), once u_c is fixed, we have two adjustable parameters (L_1 and T_1) to fit the observed stability relations between isobars. We shall show that the present data cannot be fitted very well by formula (2.18) no matter what values are assumed for L_1 and T_1.

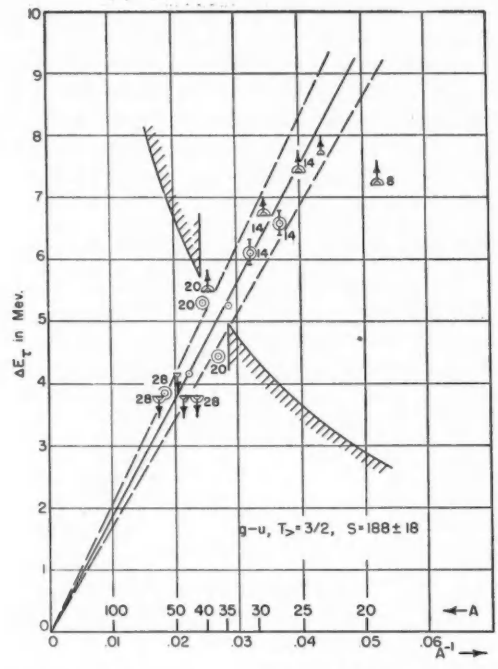

FIG. 1. Ordinate: The nuclear energy difference ΔE_T between two neighboring isobars (corrected for the Coulomb effect and for the neutron-proton mass difference) in Mev. Abscissa: The inverse of the mass number. The figure contains only those isobaric pairs which have odd mass number A $(g-u)$ and whose two members have neutron excesses $T_\zeta=\frac{1}{2}$ and $T_\zeta=\frac{3}{2}$, respectively, $(T_>=\frac{3}{2})$. Circles represent measured values, semicircles with arrows represent upper or lower limits, the arrows pointing into the allowed region. Double circles represent magic pairs (i.e., one or both of the members of the isobaric pair is magic); the particular magic number involved is indicated next to the point. The shaded region is excluded by nuclear stability data. According to the uniform model, these points are expected to fall on a straight line passing through the origin, of slope

$$s=L_1\Delta\xi+T_1\Delta\eta,$$

where L_1 and T_1 are empirical parameters to be determined, and $\Delta\xi$ and $\Delta\eta$ are given in Table I.

If a fit to the formula is nevertheless attempted, the values used by Wigner ($L_1=43$ Mev, $T_1=60$ Mev) are not the best possible ones. Lower values of L_1 and higher values of T_1 seem indicated.

3. ANALYSIS OF BINDING ENERGIES OF ISOBARIC NUCLEI

The basic assumptions of the uniform model restrict the range of validity of Eqs. (2.16)–(2.18) to a rather narrow range of mass numbers. The model does not apply to very light nuclei, since the "uniformity" assumption is almost certainly wrong there. It also cannot be expected to apply to nuclei of mass number more than 50–70. For $A\gg60$, the Coulomb repulsion between the protons becomes so significant that it can hardly be treated as a small perturbation. The basis of the group-theoretical calculation of n_+-n_- (the charge- and spin-independence of the main forces in the nucleus) then disappears, and the model fails to give correct answers. Thus, any experimental test of the uniform model must be restricted to the intermediate range of mass numbers, say between 16 and 65.

The experimental data on stability relations between isobaric nuclei are mostly derived from β-decay data. Since there are always *two* nuclei involved in a β-transition, we need a uniform notation. Let the β-transition involve the two nuclei (A, T_ζ) and (A, T_ζ'). (The *direction* of the β-decay is of no consequence here.) We shall denote the larger one of T_ζ, T_ζ' by $T_>$. Any β-transi-

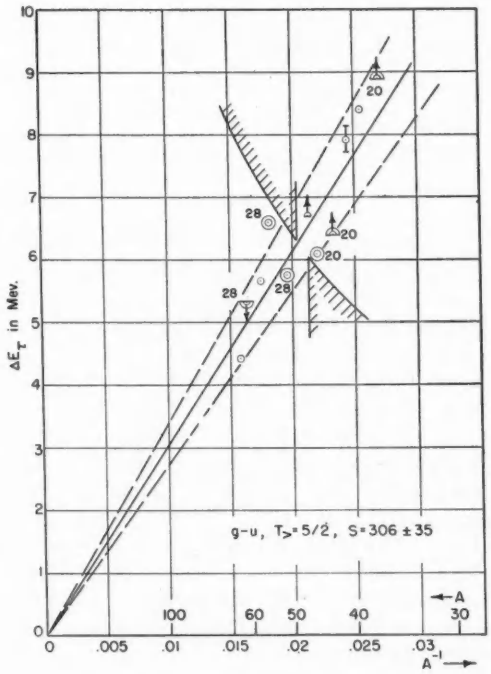

FIG. 2. Same as Fig. 1, for isobaric pairs of type $g-u$, $T_>=5/2$.

tion is specified (except for the direction of the decay) by the mass number A and by $T_>$.

For any isobaric pair of nuclei, $(A, T_>)$ and $(A, T_>-1)$, we are interested in the energy difference between their ground states. Thus, if the β-decay proceeds not to the ground state of the daughter nucleus but to an excited state, we need to know the energy of the subsequent γ-ray (or γ-rays) in addition to the maximum energy in the β-spectrum.

We have tabulated these transitions in the Appendix. For each transition, we list the mass number and atomic number, the type of decay (β^-, β^+, K capture, γ-rays), the kinetic energy release $\Delta E'$ (=the maximum kinetic energy in the β-spectrum plus the energy of the subsequent γ-rays), the Coulomb energy difference between the two nuclei [computed from Eq. (2.15) with $u_c=0.147$ Mev], and the binding energy difference corrected for the Coulomb energy difference and the neutron-proton mass difference. This latter quantity is called ΔE_τ in the tables. It represents the difference in the nuclear binding

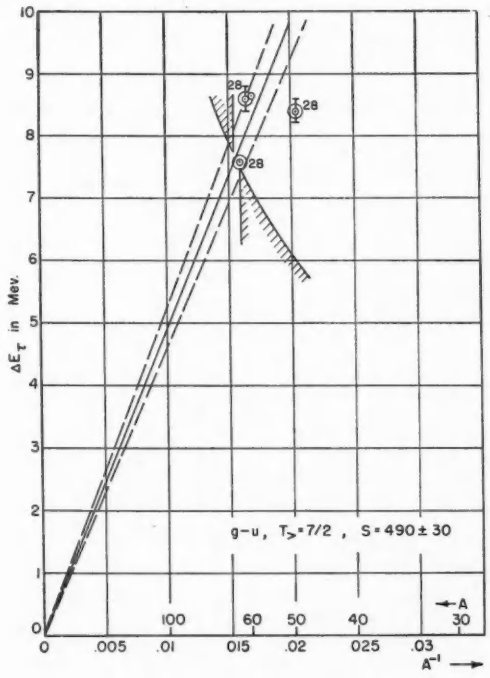

FIG. 3. Same as Fig. 1, for isobaric pairs of type $g-u$, $T_>=7/2$.

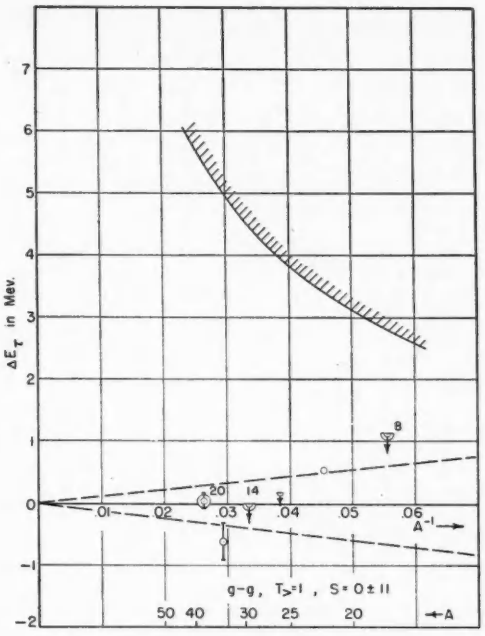

FIG. 4. Same as Fig. 1, for isobaric pairs of type $g-g$, $T_>=1$.

of the two isobars. According to Eq. (2.18), ΔE_τ is given by

$$\Delta E_\tau=(L_1\Delta\xi+T_1\Delta\eta)/A, \qquad (3.1)$$

where $\Delta\xi$ is the difference between $\xi(A, T_>)$ and $\xi(A, T_>-1)$, computed from Eq. (2.9), and $\Delta\eta$ is the corresponding quantity for η, Eq. (2.14).

We have classified the energy release data according to the "type" of the isobaric pair. By "type" we mean the value of $T_>$ together with the specification of the even-odd, even-even, or odd-odd character of the nucleus $(A, T_>)$.[14] Isobaric pairs with odd mass number A are classified as $g-u$; isobaric pairs with even mass number A are classified as $g-g$ if the nucleus $(A, T_>)$ is even-even, and are classified as $u-u$ if the nucleus $(A, T_>)$ is odd-odd.

This classification is useful because *all isobaric pairs of the same type have the same values of* $\Delta\xi$ *and* $\Delta\eta$. For example, isobaric pairs of type $u-u$ with $T_>=1$ have $\Delta\xi=13/2-5/2=4$ and $\Delta\eta=1$

[14] We emphasize once more that the *direction* of the β-decay does not concern us here. An isobaric pair $(A, T_>)$ and $(A, T_>-1)$ is classified as $g-g$ if the nucleus with the larger neutron excess, $(A, T_>)$, is an even-even nucleus, no matter whether this nucleus is β-stable or β-unstable.

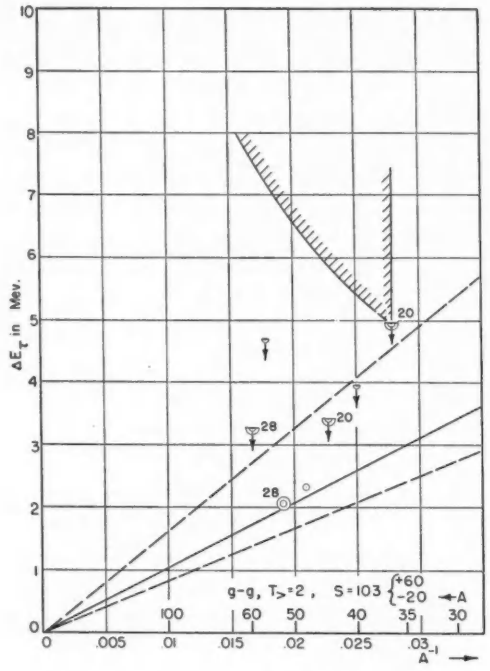

FIG. 5. Same as Fig. 1, for isobaric pairs of type $g-g$, $T_>=2$.

$-0=1$, according to Eqs. (2.9) and (2.14), respectively. Returning to Eq. (3.1), we see that ΔE_τ is expected to be proportional to A^{-1} for isobaric pairs of the same type, with proportionality constant equal to $(L_1\Delta\xi+T_1\Delta\eta)$. Thus, if we plot the experimental values of ΔE_τ against the inverse of the mass number, the points ought to fall on a straight line passing through the origin, of slope $s=L_1\Delta\xi+T_1\Delta\eta$.

Such plots are shown for the relevant types (nuclei in the mass number range $16\leq A\leq 65$) in Figs. 1–9. It is apparent from these figures that the assumptions of the uniform model are rather poorly satisfied: The points show considerable scatter about a straight line, indeed about any smooth curve.

While this is perhaps unfortunate, it is by no means unexpected and can hardly be considered as a significant failure of the model. After all, any such over-all, global approach necessarily neglects many detailed features of individual nuclei which affect their binding energies to some extent. Fluctuations of the points about any smooth curve must be expected for that reason. The most the uniform model can hope to accomplish is to give the general trend correctly, neglecting the individual fluctuations.

We have therefore adopted the following point of view: We use all the data on each figure to determine a value for the slope s of the "best" straight line (shown as a solid line on each figure), and we take into account the fluctuations by assigning an error to this slope. That is, we draw two additional lines on each figure, corresponding to our estimate of the highest and of the lowest slope still consistent with the main trend of the data, and consider their slopes as upper and lower limits of the "best" value of s, respectively. These limiting lines are shown dashed on each figure. One should then expect that these "best" slopes are represented correctly by the theoretical formula $s=L_1\Delta\xi+T_1\Delta\eta$, within their experimental errors.

In determining the "best" straight line, it is not necessary to rely exclusively upon measured energy differences between isobaric pairs. Con-

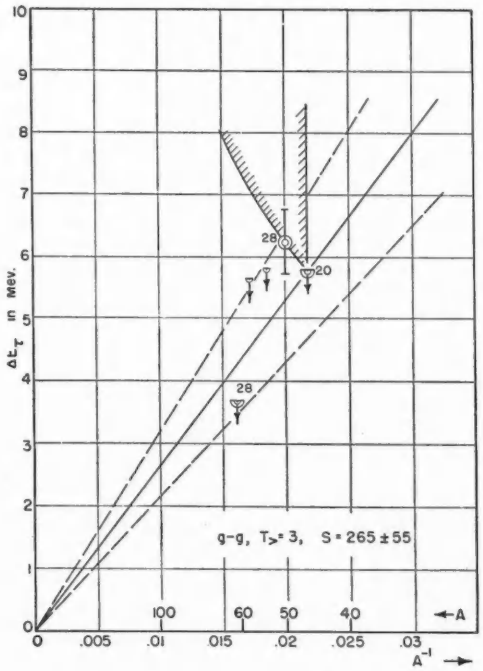

FIG. 6. Same as Fig. 1, for isobaric pairs of type $g-g$, $T_>=3$.

siderable information can be obtained from partial information about the decay scheme. For example, the information that a β^- decay with a certain maximum energy has been observed puts a lower limit on ΔE_τ, even if the subsequent γ-radiation has not been analyzed. Finally, the very information that certain nuclei are stable puts upper or lower limits, as the case may be, upon ΔE_τ. On the figures, points which represent upper or lower limits are shown with arrows, while regions of the graph which are excluded by the known stable nuclei are shown shaded. These data have also been included in the tables in the Appendix. Liberal use has been made of this additional information in order to decide upon the "best" slope.

In principle, we should exclude from these figures all isobaric pairs for which one or both of the nuclei involved are "magic." Magic pairs are shown on the figures by double circles. We see that the density of "magic numbers" is so high that about half of all the pairs are magic. Thus, exclusion of the magic pairs would have decreased the statistics very badly. However, the magic is a random effect in the sense that either of the two nuclei is as likely to be magic as the other one, other things being equal. Thus, on the average, for any magic pair with $(A, T_>)$ magic, there is another magic pair with $(A, T_>-1)$ magic. In the first case, ΔE_τ is unusually low, in the second it is unusually high. To the extent that there are equally many cases of each kind, the slope of the best straight line is not affected by the inclusion of magic pairs. The error in the slope is increased (i.e., the dotted lines have to be drawn farther away from the best line), but this is perhaps a healthy influence, since the usual tendency is to overestimate the accuracy to which the slope is known.

We would like to point out that, as an incidental by-product of this investigation, we obtain a rough estimate of the energy contributed by the various magic numbers. We see that the magic number 20 leads to an increase of the binding energy of approximately 1 Mev, and the

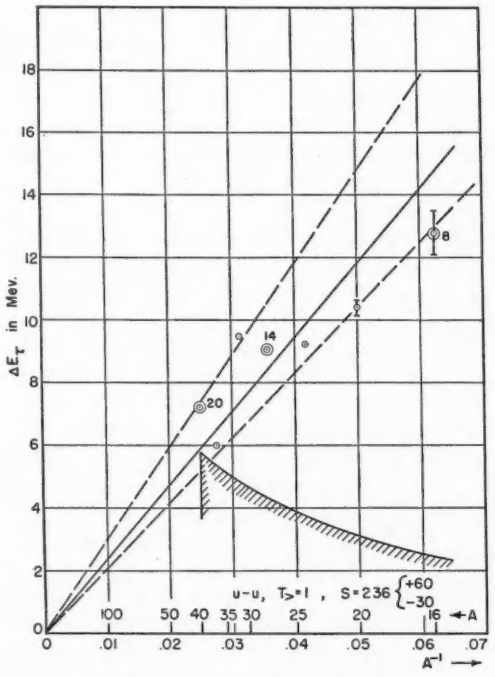

FIG. 7. Same as Fig. 1, for isobaric pairs of type $u-u$, $T_>=1$.

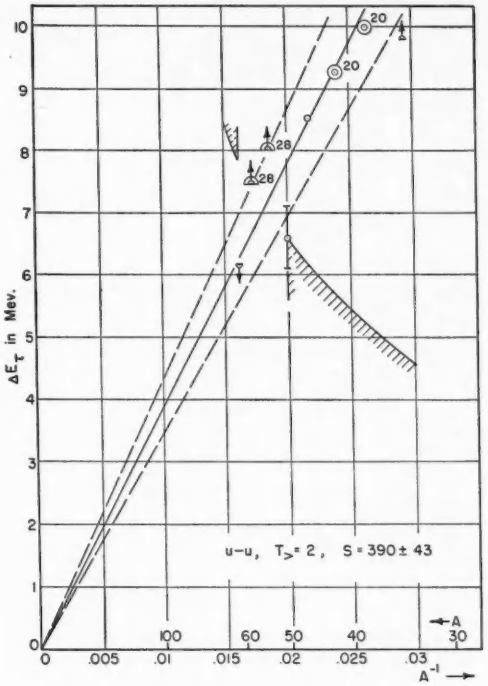

FIG. 8. Same as Fig. 1, for isobaric pairs of type $u-u$, $T_>=2$.

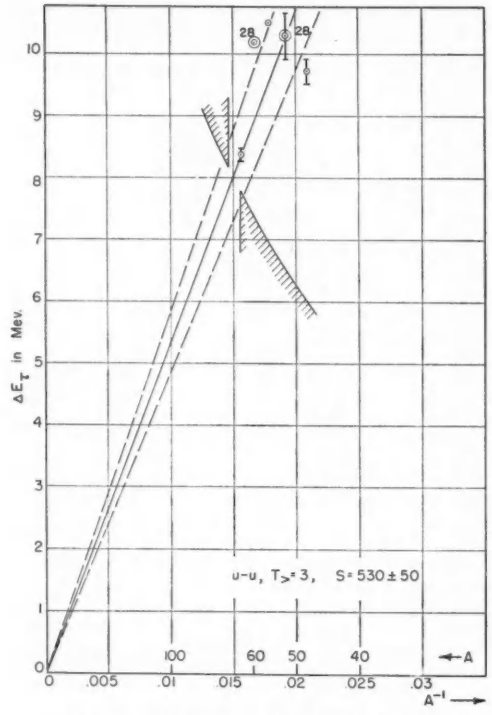

FIG. 9. Same as Fig. 1, for isobaric pairs of type $u-u$, $T_>=3$.

magic number 28 leads to a somewhat smaller increase (perhaps around $\frac{1}{2}$ Mev), whereas the so-called magic number 14 does not show up at all.

The best slopes together with their errors are compiled in Table I. In most cases it was sufficient to assign the same error to the slope in both directions. But for some of the figures the upper and lower limits on s are significantly unsymmetrical about the best value of s; in these cases the upper and lower limits are stated separately. Also given in Table I are the values of $\Delta\xi$ and $\Delta\eta$ for each type of transition.

We must now determine how we are to test these values of s for consistency with the uniform model. One particular case is simple to test: the $g-g$ transitions with $T_>=1$. The agreement is excellent. Having disposed of this case, we have 8 values of s which must be fitted by 2 independent parameters (L_1 and T_1). Since there are only two parameters to be fitted, it is possible to plot the data in such a way that these parameters are related to the slope and intercept of a straight line. To do this, we define the quantities x and y as follows:

$$\begin{aligned} &g-u \text{ transitions}, && x=3(T_>+\tfrac{3}{2}) && y=3s; \\ &g-g \text{ transitions}, && x=6T_> && y=6s; \\ &u-u \text{ transitions}, && x=2(T_>+3) && y=2s. \end{aligned} \tag{3.2}$$

Then the following equation holds for all transitions (except the $g-g$, $T_>=1$ type):

$$y=(L_1+T_1)x-6T_1. \tag{3.3}$$

Thus, a plot of y against x should be a straight line of slope L_1+T_1 and of intercept $-6T_1$. The points are shown in Fig. 10. We see that the points do not fall very well on any straight line. There is a tendency for the $g-g$ points to lie lower than the $u-u$ points by appreciable amounts. This means that *the binding energy formula of the uniform model underestimates the difference in binding energy between even-even and odd-odd nuclei.* This difficulty was already emphasized by Wigner in connection with the observed existence of triple isobars which would not be expected according to the uniform model. Figure 10 shows that the contradiction with experiment is not absolute: It is still possible to draw a straight line (the solid line shown on the figure) which passes through every point within the assigned errors. However, the appearance of the figure makes it likely that more, and more accurate, data will destroy even that bare agreement.

Let us now look at the intercept of the line on the vertical axis, which should be equal to $-6T_1$. The solid line leads to the value $T_1=109$ Mev, compared to the theoretical estimate of $T_1=60$ Mev. This is not very good agreement. To show that this is not just an accident, we have also

TABLE I. Best values of $s=L_1\Delta\xi+T_1\Delta\eta$.

Type of transition		$\Delta\xi$	$\Delta\eta$	s (in Mev)
$g-u$	$T_>=3/2$	3	1	188±18
	$T_>=5/2$	4	2	306±35
	$T_>=7/2$	5	3	490±30
$g-g$	$T_>=1$	0	0	0±11
	$T_>=2$	2	1	103(+60, −20)
	$T_>=3$	3	2	265±55
$u-u$	$T_>=1$	4	1	236(+60, −30)
	$T_>=2$	5	2	390±43
	$T_>=3$	6	3	530±50

drawn in the best line with the theoretical value of T_1 (i.e., with intercept -360 Mev). This (dashed) line gives an appreciably poorer fit to the points, missing two points altogether.

The situation is improved somewhat if we take the attitude that the uniform model breaks down for $A \gtrsim 50$ so that nuclei with $A > 50$ should be excluded from this analysis. The points at $x=15$ and $x=18$, as well as the triangle $(u-u)$ point at $x=12$, depend predominantly upon data for mass numbers between 50 and 65. If these points are omitted, the dashed straight line (with the theoretical value of T_1) goes through all the remaining points.

The best straight line on Fig. 10 corresponds to the parameters

$$L_1 = 30 \text{ Mev}, \quad T_1 = 109 \text{ Mev} \quad \text{(best line)}, \tag{3.4}$$

whereas the dashed straight line has the parameters

$$L_1 = 47 \text{ Mev (with } T_1 = 60 \text{ Mev taken from the Fermi gas model)}. \tag{3.5}$$

As is to be expected, the value $L_1=47$ is in good agreement with Wigner's value ($L_1=43$ Mev). We see that an independent adjustment of L_1 and T_1 tends to raise T_1 and lower L_1, compared to the values with T_1 fixed theoretically.

This is an unfortunate tendency from the point of view of the uniform model. The main idea behind the model is to explain the differences between energies of isobars by exchange forces together with the Pauli principle (which determines the numbers of symmetric and antisymmetric pairs). Thus, the model is constructed so as to place all the emphasis upon the *potential* energy. The kinetic energy is treated as an afterthought, as it were, by an admittedly poor approximation. This procedure can be justified *a posteriori* provided it turns out that the main contribution to the energy differences between isobars really comes from the potential energy. But this would mean $L_1 \gg T_1$, a condition which is violated already for Wigner's values, Eq. (3.5), and is badly violated for the best values, Eq. (3.4). In view of these results it appears untenable, in an explanation of isobaric energy differences with a uniform model, to place the main emphasis upon the potential energy. On the contrary, the kinetic and potential energies should

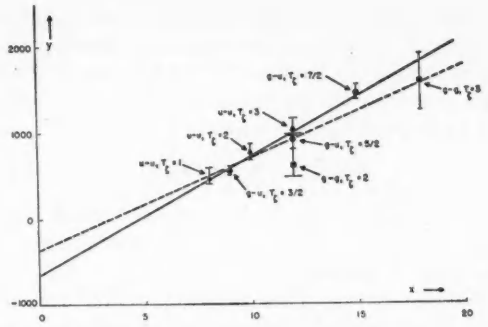

FIG. 10. Test of the formula $s=L_1\Delta\xi+T_1\Delta\eta$ for the slopes of the preceding figures. The ordinate y and abscissa x are defined by Eq. (3.2) and the points are expected to lie on a straight line given by Eq. (3.3). The fact that the points from $g-g$ transitions (the squares on the figure) fall systematically lower than the best line, while the points from $u-u$ transitions (the triangles on the figure) lie somewhat high indicates that the uniform model underestimates systematically the energy difference between even-even and odd-odd nuclei. The solid straight line represents the best fit (obtained by a modified least-squares method), while the dashed straight line represents the best fit subject to the condition that $T_1=60$ Mev, the theoretical estimate of that quantity.

be treated on an equal footing, with equally good approximations. No such treatment is available at this time.

One might object that we have completely ignored a great success of the uniform model: The zero slope for the transitions of type $g-g$, $T_>=1$ in Table I. While this is undoubtedly a success for the uniform model, it should be realized that this equality of nuclear binding for $A=4k+2$ (k integral) nuclei with $T_<=0$ and $T_<=1$ also follows from any shell model: These nuclei differ from each other merely through the fact that a proton in the top shell has been transformed into a neutron in the same shell. Thus, this success of the uniform model does not count in a comparison between the uniform model and the various shell structure (independent particle) models of the nucleus. This comparison, however, is our main point of interest, and it is for that reason that we have given little weight to this agreement between theory and experiment.

4. THE TOTAL BINDING ENERGY AND NUCLEAR SATURATION

Wigner already pointed out that the uniform model fails seriously in one respect: The model is unable to account for the magnitude of nuclear

binding energies if the parameters are adjusted to fit the binding energy *differences* between isobars. Referring back to Eq. (2.17), we see that the potential energy per nucleon is (in the limit of large mass numbers, $A \gg 1$)

$$V/A = L_1/8 - L_1'/2 + \text{order } A^{-1}. \quad (4.1)$$

This quantity has to be negative (attractive), and must be sufficiently large in absolute value to overcompensate the kinetic energy and the Coulomb energy. The analysis of the energy differences between isobars gives an experimental value of L_1. Wigner pointed out that L_1' cannot exceed L_1 if the nuclear forces obey the assumptions of the uniform model, i.e., are charge- and spin-independent and give saturation. If we use the theoretical values $T_1 = 60$ Mev, $T_0 = 13.5$ Mev for the kinetic energy terms, we get [see Eq. (3.5)] $L_1 = 47$ Mev, and hence, $L_1' \leq 47$ Mev. The potential energy per nucleon in the nucleus is then *less than* $(3/8)(47) = 17.6$ Mev. We subtract the kinetic energy per nucleon, $T_0 = 13.5$ Mev, to get a remainder of only 4.1 Mev. This would not be enough to account for the observed binding energies (of the order of 8 Mev per nucleon), even if there were no Coulomb repulsion. Actually, for $Z = 20$, $A = 40$, the Coulomb energy amounts to about 1.6 Mev per nucleon, so that the binding energy per nucleon predicted by the uniform model turns out to be *less than* 2.5 Mev. This is in clear contradiction of the experimental data.

This discrepancy becomes even worse if we use the value $L_1 = 30$ Mev obtained from the solid line of Fig. 10. With this value of L_1 and any reasonable estimate for the kinetic energy per nucleon T_0, we get no binding at all for the nucleus.

Wigner (private communication) has shown that the difficulty with the binding energy can be overcome by taking the tensor force into account. The tensor forces between the nucleons contribute an appreciable fraction of the binding energy, perhaps enough to explain the whole discrepancy. However, the inclusion of tensor forces into the calculation also has the effect of *decreasing* the expected energy difference between even-even and odd-odd nuclei. Since the conventional uniform model, using only central forces, already gives too small a value for this difference, the inclusion of tensor forces does not really help very much.

5. DISCUSSION

While the uniform model of the nucleus could muster overwhelming theoretical and experimental support some 10 years ago (with the one exception of the discrepancy in the absolute binding energies) the situation is nearly reversed today. On the theoretical side, all the assumptions underlying the uniform model are in doubt: The Wigner assumption about the nuclear forces is contradicted by the tensor force as well as by the two-nucleon scattering data and the consequent difficulties of explaining saturation by exchange forces; the uniformity assumption about nuclear structure is contradicted by the mounting evidence for shell structure in nuclei. On the experimental side, this investigation, as well as others,[10] has shown that the agreement of the binding energy formula with experiment is only moderate, and in some instances (total binding energy) rather poor. Thus, we feel that *there is no reason to consider the "success" of the uniform model as an argument against shell structure theories based upon an independent particle model of low-lying nuclear states.*

Nevertheless, we do not think that the uniform model should be abandoned altogether. In the remainder of the discussion, we would like to make some tentative suggestions for modifying the uniform model in such a way that agreement with experiment can be achieved.

Weisskopf[15] has pointed out that the evidence for nuclear shell structure is restricted entirely to low-lying (not very highly excited) nuclear states. The evidence is derived from binding energies, magnetic moments, spins, isomerism, and beta-decay, all of which are properties of low-lying states. Hurwitz and Bethe[11] have emphasized that the fast neutron-capture cross sections are inconsistent with the assumption that the entire level system of the nucleus is affected by the "magic" of the shells. Rather, they postulate the existence of a "characteristic level" whose binding energy is a smooth function of the mass number and shows no shell effects. Levels below the characteristic level are subject to shell effects, while more highly excited

[15] V. F. Weisskopf, Helv. Phys. Acta 23, 187 (1950); Science 113, 101 (1951).

levels are assumed to show no evidence of shell structure.

Hurwitz and Bethe propose that the binding energy of the characteristic level is given to a first approximation by the Weizsäcker semiempirical formula, without the "odd-even" term. We would like to suggest that the energy of the characteristic level is instead given by the "ground-state" energy formula (2.16)–(2.18) of the uniform model, which does include an "odd-even" term. The argument for this proposal is purely qualitative and very tentative only. We imagine that the highly excited states of a nucleus are so complicated in their structure that the uniform model gives a better approximation than any ordered model (e.g., independent-particle or alpha-particle model). The spectrum of these "uniform" states extends down to the "characteristic" level, which is therefore effectively the ground state as far as the uniform model is concerned. The few states below the characteristic level are highly ordered states for which the independent-particle model gives a good first approximation. Shell effects are important for those states, and the uniform model is not applicable to them.

This suggestion presupposes that the characteristic level, which separates these two regions, can be identified with some degree of accuracy. This could be done provided that there is a sharp break in the density of levels as the excitation energy increases above the characteristic level. Usually it is assumed that the density of levels $w(E)$ is a smoothly increasing function of the excitation energy E, of type

$$w(E) = C \exp(2\{aE\}^{\frac{1}{2}}), \qquad (5.1)$$

where the constants C and a are adjusted so as to reproduce the observed level densities near the ground state and near the separation energy of the first neutron (where the level density is known experimentally from a study of slow-neutron resonances). Expression (5.1) is based upon a statistical, thermodynamic treatment of the nucleus. This treatment may well be valid for levels which can be described by a uniform model. The treatment is surely invalid for states whose motion is so highly ordered that it can be described by an independent-particle (shell) model. These levels are effectively at "zero entropy" and should be excluded from thermodynamic considerations. While we have no clear idea of the behavior of the level density in this zero entropy region, it is surely not too far from a uniform density of levels, at least qualitatively. We are therefore led to consider the following level density formula:

$$w(E) = C \quad \text{for} \quad 0 \leq E \leq E_c,$$
$$w(E) = C \exp[2\{a(E-E_c)\}^{\frac{1}{2}}] \quad \text{for} \quad E \geq E_c, \qquad (5.2)$$

where E_c is the excitation energy of the characteristic level and C and a are adjustable constants

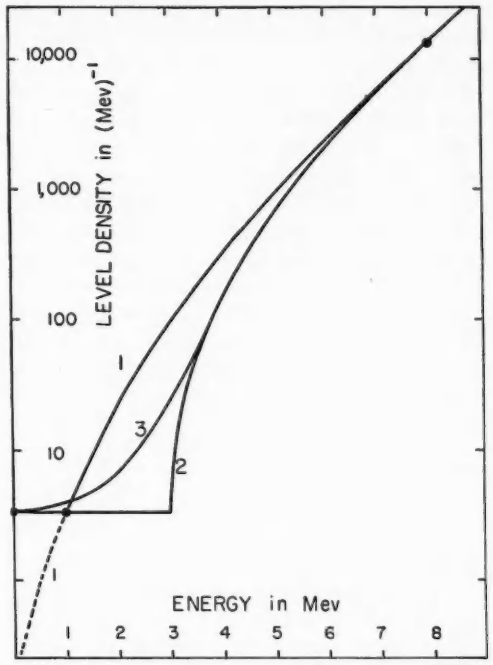

FIG. 11. Schematic picture of the level density $w(E)$ as a function of the excitation energy E. Curve 1 shows the usually assumed behavior, formula (5.1), with constants fitted at $E=1$ Mev and at $E=8$ Mev. The very sharp slope of $w(E)$ near $E=0$ is usually considered spurious and is therefore ignored; the extrapolated value of $w(E)$ at $E=0$ would be $w(0)=3.5\times10^{-2}$ (Mev)$^{-1}$, an impossible value. Curve 2 shows the result of formula (5.2); the energy of the characteristic level was assumed to be $E_c=3$ Mev; the sharp discontinuity in slope at this energy is undoubtedly not realized in nature. Curve 3 shows the result of formula (5.3); this is qualitatively the same as formula (5.2), except that the discontinuity has been smoothed out arbitrarily. This figure shows that even though the constants in formulas (5.1) and (5.3) have been fitted so as to give the same level densities near the ground state and at 8 Mev, these two formulas predict quite different behavior of the level density in the intermediate region.

which are to be used to fit the observed level densities near the ground state and near the separation energy of the first neutron.

Formula (5.2) has one qualitative defect: There is an extremely sharp break in the level density at the energy of the characteristic level. Indeed, the derivative dw/dE is zero if E approaches E_c from below, infinite if E approaches E_c from above. There is no reason to believe that the distinction between the two regions (above and below E_c) is that sharp. We therefore need a formula which approaches a constant value for $E \ll E_c$, and approaches the value given by Eq. (5.2) for $E \gg E_c$. The following is suggested as a plausible interpolation formula:

$$w(E) = C \exp\left\{2\left[\frac{a(E-E_c)}{1-\exp[-a(E-E_c)]}\right]^{\frac{1}{2}}\right\}. \quad (5.3)$$

The differences between formulas (5.1), (5.2), and (5.3) are illustrated in Fig. 11. In this figure the level densities are adjusted to the same value at $E=8$ Mev, and Eqs. (5.2) and (5.3) are adjusted to a common value at $E=0$. Since it has been the conventional practice to adjust the constants in Eq. (5.1) to fit the density of low-lying levels at $E=1$ Mev rather than at $E=0$, we have adjusted the constants in Eq. (5.1) so that this function has the same value at $E=1$ Mev as the other two functions assume at $E=0$. The energy of the characteristic level was taken arbitrarily to be $E_c=3$ Mev.

We remark that the conventional practice of ignoring the sharp energy dependence of Eq. (5.1) near $E=0$ conceals a difficulty: according to its derivation, the exponent $2\{aE\}^{\frac{1}{2}}$ represents an entropy, and its energy derivative $\{a/E\}^{\frac{1}{2}}$ represents the inverse of the temperature of the nucleus. Thus, if we replace the theoretical level density Eq. (5.1) near $E=0$ by a more slowly varying function (which is in better agreement with experiment), we are decreasing the energy derivative of the entropy drastically. Since experimentally the level density near zero energy is a *slowly* varying function of E, the thermodynamic interpretation of the nuclear level structure leads to the paradoxical result that zero energy of the nucleus corresponds to a very *high* temperature. The paradox is resolved of course by remarking that thermodynamic considerations cannot be applied to these low-lying levels, so that the exponent should not be interpreted as a nuclear entropy in that region.

There are various indications that Eq. (5.1) is not adequate to describe the level spectrum of actual nuclei. The most direct experiments on level densities are probably those of Gugelot[16] on the spectra of secondary neutrons from nuclear reactions. Gugelot adjusts both C and a in Eq. (5.1) to fit his data near the separation energy of the first neutron. Then he observes that the extrapolated value of the level density near $E=0$ is much too low, i.e., the extrapolated energy spacing of the low-lying levels is very much larger than the observed one. It is apparent that a level density formula such as Eq. (5.3) would give a better fit. However, we do not believe that we are justified in fitting Eq. (5.3) to Gugelot's data at this time, since it has not been demonstrated experimentally that the sharp break in $w(E)$ at $E=E_c$ predicted by Eq. (5.3) really exists in nature.

The excitation energy E_c of the characteristic level is *not* expected to be a smooth function of N and Z. Energy E_c is the difference between the binding energy of the ground state of the nucleus (which is subject to shell effects) and the binding energy of the characteristic level. Only the latter quantity is expected to be a smooth function of N and Z, given by Eqs. (2.16)–(2.18). In particular, we expect E_c to be abnormally large (perhaps 4 to 6 Mev) in "magic" nuclei, compared to its value (perhaps 2 to 4 Mev) in ordinary nuclei.

Nothing can be said about either the validity of this whole approach, or the numerical values of E_c, until much more is known about level spectra of medium-weight and heavy nuclei. We have given Eq. (5.3) not because we are sure it is correct, but because we believe that Eq. (5.1) is incorrect and some alternative formula is necessary, if only as a stopgap until the level spectra become better known. It is perhaps not sufficiently realized just how important the level density formula is for many of the common estimates in the theory of nuclear reactions, such as the ratio of widths for emissions of different particles. In general, formula (5.3) favors emission of neutrons from the compound nucleus over

[16] P. C. Gugelot, Phys. Rev. **81**, 51 (1951).

TABLE A-1: $g-u$, $T_>=3/2$.

A	Z	Rad.	Magic numbers involved	$\Delta E'$	$\lvert\Delta C\rvert$	ΔE_τ	A^{-1}	References
19	8	β^-, γ	8	$\geqslant 4.5$	3.54	$\geqslant 7.2$	0.0526	
23	10	β^-		4.3	4.16	7.7	0.0435	
25	11	β^-, γ	14	3.7	4.44	7.4	0.0400	
27	12	β^-, γ	14	2.64 ± 0.2	4.72	6.58 ± 0.2	0.0370	
29	13	β^-, γ	14	$\geqslant 2.5$	5.00	$\geqslant 6.7$	0.0345	
31	14	β^-	14	1.6 ± 0.2	5.26	6.1 ± 0.2	0.0323	
35	16	β^-		0.168	5.77	5.16	0.0286	
37	18	K	20	0.816	6.02	4.42	0.0270	(18)
39	19	st.	20		6.27	$\geqslant 5.49$	0.0257	
41	20	K	20	0.44	6.51	5.29	0.0244	(18)
43	21	β^+, γ	20	$\geqslant 1.1$	6.73	$\leqslant 3.8$	0.0233	
45	22	β^+, γ		1.00	6.96	4.16	0.0222	(19), (20)
49	24	β^+, γ		$\geqslant 1.45$	7.42	$\leqslant 4.17$	0.0204	
55	27	β^+, γ	28	2.43	8.06	3.83	0.0182	
57	28	β^+, γ	28	$\geqslant 2.7$	8.28	$\leqslant 3.8$	0.0175	

TABLE A-2: $g-u$, $T_>=5/2$.

A	Z	Rad.	Magic numbers involved	$\Delta E'$	$\lvert\Delta C\rvert$	ΔE_τ	A^{-1}	References
37	16	β^-, γ	20	$\geqslant 4$	5.67	$\geqslant 8.9$	0.0270	
39	17	β^-		3.31	5.91	8.44	0.0256	(21)
41	18	β^-, γ		2.55 ± 0.2	6.16	7.93 ± 0.2	0.0244	
43	19	β^-, γ	20	$\geqslant 0.81$	6.41	$\geqslant 6.44$	0.0233	
45	20	β^-	20	0.25	6.63	6.10	0.0222	
47	21	β^-, γ		$\geqslant 0.6$	6.87	$\geqslant 6.7$	0.0213	
49	22	st.			7.10	$\leqslant 6.32$	0.0204	
51	24	K	28	0.750	7.31	5.78	0.0196	(18), (22)
55	26	K	28	0.38	7.76	6.60*	0.0182	(18)
57	27	β^+, γ		0.39	7.97	5.68	0.0175	
61	29	β^+	28	$\geqslant 1.22$	8.38	$\leqslant 5.36$	0.0164	
63	30	β^+, K, γ		2.36	8.60	4.44	0.0159	

TABLE A-3: $g-u$, $T_>=7/2$.

A	Z	Rad.	Magic numbers involved	$\Delta E'$	$\lvert\Delta C\rvert$	ΔE_τ	A^{-1}
49	21	β^-	28	2.4 ± 0.2	6.77	8.4 ± 0.2*	0.0204
59	26	β^-, γ		1.56	7.88	8.66	0.0169
61	27	β^-	28	1.3 ± 0.2	8.09	8.6 ± 0.2	0.0164
63	28	β^-	28	0.063	8.30	7.58	0.0159

TABLE A-4: $g-g$, $T_>=1$.

A	Z	Rad.	Magic numbers involved	$\Delta E'$	$\lvert\Delta C\rvert$	ΔE_τ	A^{-1}	References
18	9	β^+, γ?	8	$\geqslant 0.64$	3.61	$\leqslant 1.15$	0.0556	
22	11	β^+, γ		1.85	4.22	0.57	0.0455	
26	13	β^+		$\geqslant 2.8$	4.78	$\leqslant 0.2$	0.0385	
30	15	2^+	14	$\geqslant 3.5$	5.31	$\leqslant 0$	0.0333	
34	17	β^+, γ		4.6 ± 0.3	5.84	0.6 ± 0.3	0.0294	(23)
38	19	β^+, γ	20	4.6 ± 0.1	6.33	0.07 ± 0.1	0.0263	

TABLE A-5: $g-g$, $T_>=2$.

A	Z	Rad.	Magic numbers involved	$\Delta E'$	$\lvert\Delta C\rvert$	ΔE_τ	A^{-1}
36	16	st.	20		5.72	$\leqslant 4.94$	0.0278
40	19	K, γ		$\geqslant 1.45$	6.20	$\leqslant 3.97$	0.0250
44	21	β^+, K, γ	20	$\geqslant 1.45$	6.70	$\leqslant 3.45$	0.0227
48	23	β^+, K, γ		3.03	7.16	2.33	0.0208
52	25	β^+, K, γ	28	3.72	7.60	2.08	0.0192
56	27	β^+, γ		$\geqslant 1.5$	8.02	$\leqslant 4.7$	0.0179
60	29	β^+, γ	28	$\geqslant 3.3$	8.45	$\leqslant 3.3$	0.0167

TABLE A-6: $g-g$, $T_>=3$.

A	Z	Rad.	Magic numbers involved	$\Delta E'$	$\lvert\Delta C\rvert$	ΔE_τ	A^{-1}
46	20	st.	20		6.59	$\leqslant 5.81$	0.0217
50	23	st.	28	0 ± 0.5	7.06	6.3 ± 0.5	0.0200
50	22	st.					
54	25	K		$\geqslant 0.835$	7.49	$\leqslant 5.84$	0.0185
58	27	β^+, γ		$\geqslant 0.47$	7.93	$\leqslant 5.66$	0.0172
62	29	β^+, γ	28	$\geqslant 2.9$	8.35	$\leqslant 3.7$	0.0161

TABLE A-7: $u-u$, $T_>=1$.

A	Z	Rad.	Magic numbers involved	$\Delta E'$	$\lvert\Delta C\rvert$	ΔE_τ	A^{-1}
16	7	β^-, γ	8	10.3 ± 0.7	3.28	12.8 ± 0.7*	0.0625
20	9	β^-, γ		7.3 ± 0.2	3.92	10.4 ± 0.2*	0.0500
24	11	β^-, γ		5.53	4.51	9.26	0.0417
28	13	β^-, γ	14	4.81 ± 0.1	5.06	9.09 ± 0.1	0.0357
32	15	β^-		1.71	5.58	9.51*	0.0313
36	17	β^-		0.713	6.07	6.00	0.0278
40	19	β^-, γ	20	1.4 ± 0.1	6.57	7.2 ± 0.1*	0.0250

TABLE A-8: $u-u$, $T_>=2$.

A	Z	Rad.	Magic numbers involved	$\Delta E'$	$\lvert\Delta C\rvert$	ΔE_τ	A^{-1}
34	15	β^-		$\geqslant 5.1$	5.46	$\geqslant 9.8$	0.0294
38	17	β^-, γ	20	4.81	5.98	10.01	0.0263
42	19	β^-, γ	20	3.58	6.47	9.27	0.0238
46	21	β^-, γ		2.38	6.93	8.53	0.0217
50	23	st.		0 ± 0.5	7.36	6.6 ± 0.5	0.0200
50	24	st.					
54	25	β^-, γ	28	$\geqslant 1.0$	7.81	$\geqslant 8.0$	0.0185
58	28	st.	28		8.23	$\geqslant 7.45$	0.0172
62	30	β^+, K, γ		$\geqslant 0.665$	8.64	$\leqslant 6.17$	0.0161

TABLE A-9: $u-u$, $T_>=3$.

A	Z	Rad.	Magic numbers involved	$\Delta E'$	$\lvert\Delta C\rvert$	ΔE_τ	A^{-1}	References
48	21	β^-, γ		3.64 ± 0.2	6.83	9.7 ± 0.2*	0.0208	(24)
52	23	β^-, γ	28	3.8 ± 0.4	7.28	10.3 ± 0.4	0.0192	
56	25	β^-, γ		3.63 ± 0.1	7.71	10.56 ± 0.1	0.0179	
60	27	β^-, γ	28	2.81	8.15	10.18*	0.0167	
64	29	β^-		0.57 ± 0.1	8.57	8.36 ± 0.1	0.0156	

emission of charged particles to a larger extent than does formula (5.1), the numerical difference between the two being quite appreciable. The differences become even more pronounced if the excitation energy of the compound nucleus is high enough so that secondary and tertiary reactions become possible. The ratio of the widths for

two-neutron emission, and emission of a neutron and a proton, for example, is extremely dependent upon what is assumed for the level density in the various residual nuclei.

At the very worst, Eq. (5.3) is likely to give a better approximation to the truth than Eq. (5.1) merely because it contains one more adjustable parameter (the excitation energy of the characteristic level).

ACKNOWLEDGMENTS

The authors wish to express their sincere thanks to Professor Wigner of Princeton University and Professor Weisskopf of Massachusetts Institute of Technology for many extremely helpful suggestions and criticisms of this work. We would also like to express our appreciation for the excellent compilation of nuclear data by the National Bureau of Standards[17] which greatly shortened our labor.

APPENDIX: TABLES OF TRANSITION ENERGIES

These tables give the energy differences between isobars, organized according to the "type" of the isobaric pair. Symbol $T_>$ is the value of the neutron excess $T_z=\frac{1}{2}(N-Z)$ for the isobar with the *larger* neutron excess (independently of whether this is the radioactive or the stable member of the isobaric pair in question). Symbol $g-u$ indicates that the mass number is odd; $g-g$ indicates that the isobar $(A, T_>)$ is an even-even nucleus; $u-u$ indicates that the isobar $(A, T_>)$ is an odd-odd nucleus.

In each table, the first two columns give the mass number and atomic number of the nucleus from which the relevant information is derived (usually the radioactive nucleus). The next two columns give the type of radioactivity observed (st = stable) and the magic numbers involved, if any. The fifth column gives the kinetic energy release $\Delta E'$ in the radioactive decay. This quantity is defined as the maximum *kinetic* energy of the decay electron (or positron) plus the energy subsequently emitted in the form of γ-rays. In the case of K capture (where the information is derived from nuclear reaction data rather than from β-decay data) $\Delta E'$ is the total energy difference and is equal to the energy given to the neutrino. The fifth column gives the Coulomb energy difference $|\Delta C|$ computed from Eq. (2.18) with $4u_c=0.590$ Mev (this value is derived from the energy differences between mirror nuclei). The sixth column gives ΔE_τ, the nuclear energy change corrected for the Coulomb energy difference and for the neutron-proton mass difference. According to the uniform model, ΔE_τ is given by Eq. (3.1) and is inversely proportional to A. Column 8 therefore contains the value of A^{-1}, and Figs. 3.1 to 3.9 contain plots of ΔE_τ *versus* A^{-1}. A star is attached to energies ΔE_τ which are considerably (more than about 1 Mev) removed from the best straight line on the corresponding figure. Some of these deviations can be explained by magic numbers (see column 4), and in all these cases the deviation is in the right direction, i.e., the magic nucleus is unusually stable. We see that the magic number 14 is not very potent, but the magic numbers 20 and 28 lead to an increase in binding energy of between 1 and 2 Mev. There are some starred cases in which neither of the two nuclei is magic. Some of these deviations may be due to experimental errors; the others are not explained by either the uniform model or by the shell model. The last column contains those references which were not included in the *Compilation of Nuclear Data* by the U. S. Bureau of Standards.[17] Inclusion of all the relevant references would have lengthened the tables unduly.

[17] Way, Fano, Scott, and Thew, "Nuclear Data," Circular of the U. S. National Bureau of Standards 499, September 1, 1950.

The adventurer is an individualist and an egotist, a truant from obligations. His road is solitary, there is no room for company on it. What he does, he does for himself.—WILLIAM BOLITHO.

NOTES AND DISCUSSION

Momentum Thrust of a Rocket

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THE common methods for deriving an expression for the thrust developed by a rocket motor vary in complexity from simply writing down the result[1] to the derivation of an auxiliary general principle of mechanics.[2] Some writers utilize a theorem from hydrodynamics[3,4] or an integration of the forces exerted over the exterior and interior of the rocket,[5] including in the integration the reaction force due to the burned propellant leaving the rocket. The authors feel that it is desirable to derive the momentum thrust (that thrust due to the acceleration of the jet mass) directly from classical mechanics and to consider any other forces, such as pressure differences over the rocket surface, as external forces. This opinion does not imply that the derivations referred to above are incorrect, but rather that they are open to one or more of the following objections:

(1) They require special study of an auxiliary theorem, such as the hydrodynamic theorem or the auxiliary principle of mechanics.

(2) Some of the assumptions may be open to question; for example, the application of the hydrodynamic theorem assumes that the flow of the surrounding medium is not disturbed and that the jet mass does not mix with the surrounding medium.

(3) They do not clearly present the mechanics of the problem; this is particularly evident if the result is written down immediately with little or no attempt at a derivation. To be more explicit, consider the result as obtained in reference 1. It is concluded as being almost immediately obvious that the force on a rocket due to the jet is

$$\mathbf{F}=-\dot{m}\mathbf{v}_e,$$

where $\dot{m}$ is the rate at which the mass is leaving the rocket and $\mathbf{v}_e$ is the velocity of the jet relative to the rocket. However, if an unwary student applies Newton's second law to the rocket to obtain

$$\mathbf{F}=-\dot{m}\mathbf{v}_e=d(M\mathbf{v})/dt,$$

where M is the mass of the rocket and $\mathbf{v}$ is its velocity, and, sensibly, attempts to account for the variation of rocket mass in the differentiation, he will obtain an incorrect expression.

(4) Velocities are sometimes determined with respect to more than one coordinate system; in particular, the rocket velocity $\mathbf{v}$ is determined relative to a fixed coordinate system, while the jet velocity $\mathbf{v}_e$ is determined relative to the rocket, with no discussion presented.

(5) The relative jet velocity is considered constant in most derivations, and some do not include a variation in absolute jet velocity as the rocket moves.

It is true that some of the above methods have particular advantages. For example, the method which uses the hydrodynamic theorem includes in the result the pressure thrust as well as the momentum thrust. On the other hand, the derivation which follows has the advantages that (1) it is direct; (2) it applies to the general case in that it considers variations in all quantities which can change; (3) it attempts to present a clear physical picture.

We shall apply a principle from mechanics which states that the time rate of change of the total momentum of a system of particles is equal to the vector sum of the external forces on the system. It should be emphasized that this principle pertains to a system which always contains the same particles. We apply the theorem to the system which is defined to be that group of particles which comprise the total mass of the rocket and fuel at a time t. Letting $\mathbf{G}$ represent the momentum of this system,

$$d\mathbf{G}/dt=\Sigma\mathbf{F},$$

in which $\Sigma\mathbf{F}$ is the sum of the external forces.

We define the following quantities: M = total mass of rocket at time t; $\mathbf{v}$ = velocity of rocket at time t, measured relative to a fixed coordinate system; $\mathbf{v}_j$ = velocity of jet as it crosses the exit plane of the rocket nozzle at time t, measured relative to a fixed coordinate system; $\mathbf{v}_e$ = velocity of jet as it crosses the exit plane of the nozzle at time t, measured relative to the rocket nozzle; and Δm = propellant mass which leaves the rocket during a time interval Δt;

At the time t, the total mass of the system is moving with the speed of the rocket, so

$$\mathbf{G}(t)=M\mathbf{v}.$$

At the time $t+\Delta t$, a quantity of mass Δm has left the rocket at an average velocity $\mathbf{v}_j+\frac{1}{2}\Delta\mathbf{v}_j$, so that

$$\mathbf{G}(t+\Delta t)=(M-\Delta m)(\mathbf{v}+\Delta\mathbf{v})+\Delta m(\mathbf{v}_j+\tfrac{1}{2}\Delta\mathbf{v}_j).$$

By definition of the derivative,

$$d\mathbf{G}/dt=\lim_{\Delta t\to 0}[\mathbf{G}(t+\Delta t)-\mathbf{G}(t)]/\Delta t$$

$$=\lim_{\Delta t\to 0}[(M\mathbf{v}+M\Delta\mathbf{v}-\mathbf{v}\Delta m-\Delta m\Delta\mathbf{v}+\Delta m\mathbf{v}_j+\tfrac{1}{2}\Delta m\Delta\mathbf{v}_j)-M\mathbf{v}]/\Delta t.$$

It is convenient to express the final result for momentum thrust in terms of $\mathbf{v}_e$, the jet velocity measured relative to the rocket nozzle, since this quantity is usually known to the designer of rockets. Toward this end, since, vectorially,

$$\mathbf{v}_j=\mathbf{v}+\mathbf{v}_e$$

and

$$\Delta\mathbf{v}_j=\Delta\mathbf{v}+\Delta\mathbf{v}_e,$$

it follows by a straightforward substitution that

$$d\mathbf{G}/dt=\lim_{\Delta t\to 0}(M\Delta\mathbf{v}-\tfrac{1}{2}\Delta m\Delta\mathbf{v}+\Delta m\mathbf{v}_e+\tfrac{1}{2}\Delta m\Delta\mathbf{v}_e)/\Delta t.$$

Note that some of the terms cancel before proceeding to the limit, and the remaining terms can be interpreted as follows: The first term claims that the rocket has suffered a change in momentum $M\Delta\mathbf{v}$, while the second term

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counterclaims that not quite all of M was accelerated by the amount Δv since some mass had left the system. The third term describes the momentum change of the jet mass, while the last term corrects this for a possible change in relative jet velocity. Proceeding to the limit,

$$dG/dt = Mdv/dt + dm/dt v_e,$$

in which the second and fourth terms have been discarded as terms which approach zero in the limit.

The final result is, as is usually obtained;

$$Mdv/dt = -dm/dt v_e + \Sigma F,$$

where dG/dt has been replaced by ΣF. The minus sign indicates that the rocket acceleration is opposite to the direction of v_e. The pressure thrust, $A_e(P_3 - P_e)$, which may exist because of a difference between the pressure P_e of the jet at the nozzle exit, and the pressure P_3 of the surrounding medium, is included in ΣF. v_e is the actual relative jet velocity. A theoretical expression for v_e can be obtained in terms of the conditions in the rocket motor combustion chamber and the pressure at the nozzle exit.[6]

The expression is valid for all time, since no restriction was placed upon t. It should be borne in mind, however, that the total mass M is a function of time, so that the acceleration of the rocket body is not constant, even if $\dot{m}$, v_e, and ΣF are constant.

[1] Seifert, Mills, and Sommerfield, Am. J. Phys. 15, 1 (1947).
[2] Rosser, Newton, and Gross, *Mathematical Theory of Rocket Flight* (McGraw-Hill Book Company, Inc., New York, 1947), p. 6.
[3] M. J. Zucrow, *Principles of Jet Propulsion and Gas Turbines* (John Wiley and Sons, Inc., New York, 1948), pp. 38, 491.
[4] G. P. Sutton, *Rocket Propulsion Elements* (John Wiley and Sons, Inc., New York, 1949), p. 12.
[5] Reference 3, pp. 38, 491.
[6] Reference 4, p. 45, Eq. (3–14).

A Method for Deriving Various Formulas in Electrostatics and Electromagnetism Using Lagrange's Trigonometric Identities

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IN many textbooks in college physics authors very often do not derive formulas in electrostatics dealing with continuous distribution of charges and in electromagnetism dealing with the fields set up by electric currents because a summation process, integral calculus, is required. With calculus such derivations are easily carried out but without it they are rather difficult.

The purpose of this paper is to present a method for computing various formulas in electrostatics and electromagnetism using the trigonometric summations $\sum_1^N \cos n\theta$ and $\sum_1^N \sin n\theta$ known as Lagrange's trigonometric identities.

Throughout this paper all angles will be divided into a large finite number of equal units. As the size of the unit θ is arbitrary, as small as one pleases, and hence the number of units very large, the trigonometric summations can be approximated as shown below:

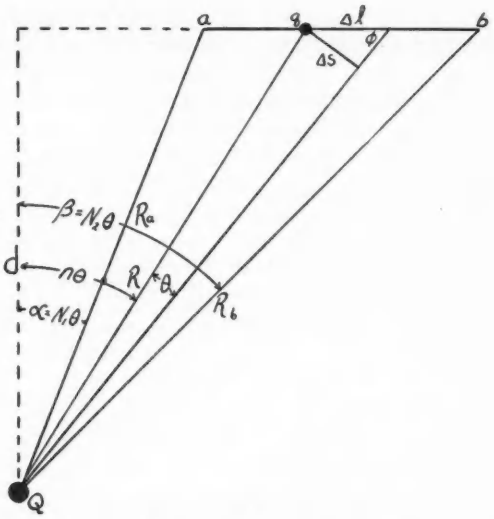

FIG. 1. Calculating the work required to bring a charge q from a to b in the field of a point charge Q.

Summation	Exact value	Approximate value
$\sum_1^N \cos n\theta$	$-\frac{1}{2}+\frac{\sin(N+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}$	$-\frac{1}{2}+\frac{\sin N\theta}{\theta}$
$\sum_1^N \sin n\theta$	$\frac{\cot\frac{1}{2}\theta}{2}-\frac{\cos(N+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}$	$\frac{1-\cos N\theta}{\theta}$.

Since

$$\sum_{N_1}^{N_2} \cos n\theta = \sum_1^{N_2} \cos n\theta - \sum_1^{N_1} \cos n\theta$$

and

$$\sum_{N_1}^{N_2} \sin n\theta = \sum_1^{N_2} \sin n\theta - \sum_1^{N_1} \sin n\theta,$$

the above equations can be approximated to

$$\sum_{N_1}^{N_2} \cos n\theta = \frac{\sin N_2\theta - \sin N_1\theta}{\theta}, \tag{1}$$

$$\sum_{N_1}^{N_2} \sin n\theta = \frac{\cos N_2\theta - \cos N_1\theta}{\theta}. \tag{2}$$

The application of the method will be illustrated by the following two examples.

1. To calculate the work required to move a positive charge q from a point a to a point b in the region of a positive point charge Q.

The force acting on q at a distance R from Q is

$$F = qQ/R^2.$$

The work done against the electrical force F in a displacement Δl is

$$\Delta W = -F\cos\phi\Delta l = -qQ\cos\phi\Delta l/R^2.$$

It can be seen from Fig. 1 that

$$\Delta l = \Delta S/\sin\phi = R\theta/\cos n\theta$$

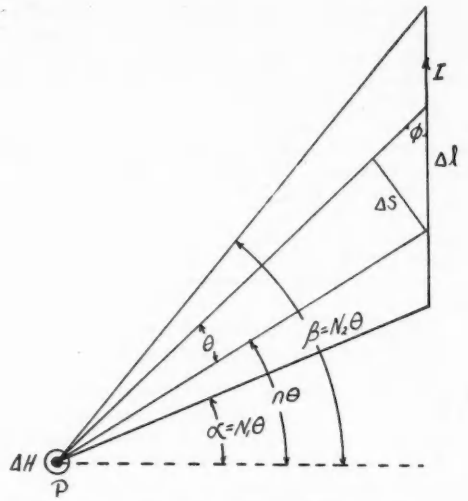

FIG. 2. Calculating the magnetic field of a piece of wire carrying a current I. The magnetic field is toward the reader and is represented by ⊙.

and

$$R=d/\cos n\theta.$$

Hence, the work becomes

$$W=-qQ\theta\sin n\theta/d.$$

The total work required to move q from a to b is

$$W=-qQ\theta/d\sum_{N_1}^{N_2}\sin n\theta.$$

When Eq. (2) is used,

$$W=\frac{qQ(\cos\beta-\cos\alpha)}{d}.$$

Since

$$R_a=\cos\alpha/d,\quad R_b=\cos\beta/d,$$

then

$$W=qQ\left(\frac{1}{R_b}-\frac{1}{R_a}\right).$$

Since the quotient W/q is defined as the potential difference between a and b, the potential at a point at a distance R from Q relative to a point at infinity is $V=Q/R$.

2. To calculate the magnetic field at a point P of a piece of straight wire carrying a current I.

According to Ampère's theorem, a current element $I\Delta l$ (Fig. 2) produces a field ΔH at P given by

$$\Delta H=I\Delta l\sin\phi/R^2,$$

where I is the current, R the distance from the element to P, and ϕ the angle between $I\Delta l$ and R.

It can be seen from the Fig. 2 that

$$\Delta l=R\theta/\cos n\theta,\quad R=d/\cos n\theta.$$

Hence, the contribution ΔH due to the element becomes

$$\Delta H=I\theta\cos n\theta/d.$$

Since the contribution of all current elements is in the same direction, the total field at P is

$$H=I\theta/d\sum_{N_1}^{N_2}\cos n\theta.$$

When Eq. (1) is used,

$$H=I(\sin\beta-\sin\alpha)/d.$$

If the wire is infinitely long, $\beta=\pi/2$ and $\alpha=-\pi/2$, then $H=2I/d$. This formula is generally called the Biot-Savart law.

It has thus been shown that for various field problems in electrostatics and electromagnetism it is possible to derive formulas by means of two trigonometric identities. It is advantageous to make derivations by a method closely paralleling the calculus approach, since it will mesh better with any work the student may take later. The derivations lend themselves to be used in a course of college physics, and in the light of the author's experience it may prove to be a valuable teaching aid.

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LETTERS TO THE EDITOR

Complementarity in the Life Sciences

MR. POLLARD'S interesting article, "Significance of Complementarity in the Life Sciences,"[1] has prompted me to make a few remarks.

I think we must be warned that verbal interpretations such as "complementarity" and "indeterminacy" which are made directly from an abstract mathematics, because the phenomena which this mathematics describe are untenable to five-sense experience, really tell us more about ourselves than about the nature of the phenomena. Because words are rooted in five-sense experience they may not be appropriate or meaningful for phenomena beyond our direct perception, and any attempted interpretations may reveal certain predispositions in our own natures. In some respects the mathematical equation becomes a projective device, not unlike the psychologist's Rorschach test (an abstract series of ink forms) from which verbal interpretations give insight into personality.

Though verbal interpretation from abstract mathematics helps to satisfy our psyches more than the mathematics can (because the language of the psyche is and can

only be the verbal language), there is no necessity that the essential intelligence of the mathematics is verbally translatable. After all, we use mathematics precisely because of the fact that words are inevitably handcuffed to certain unconscious associations while mathematical symbols are not and hence are free to transcend the limitations of whatever predispositions are inherent in human nature.

It should not be surprising to find that in other fields (history, biology, psychology) the organization of verbal statements divides along a cleavage similar to that found in physics. This merely says that the physicist, historian, biologist, psychologist, is the same kind of being. If it is true that the verbal concepts, "complementarity" and "indeterminacy," are applicable to these other fields, this informs us of something interesting about the nature of the mind, and it is possible that some work may be shortcutted in advancing the other fields by properly orienting the biologist, historian, etc. But this must suppose, of course, that the particular phrasing of "complementarity" and "indeterminacy" is general enough to apply elsewhere.

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[1] W. G. Pollard, Am. J. Phys. **20**, 281 (1952).

Complementarity in the Life Sciences

PROFESSOR Fein's comments arise out of a point of view which looks upon notions of "complementarity" and "indeterminacy" in physics as being abstract mathematical concepts. From this viewpoint it then appears that what I am suggesting in my paper is just another way to put the life sciences on a rigorous mathematical basis, or to express their content within the framework of a mathematical theory. This, however, was not at all the objective I had in mind. Actually, I think it is erroneous to regard either complementarity or indeterminacy in physics as interpretations of phenomena "made directly from an abstract mathematics" as Professor Fein suggests. Both of these concepts have to do with the way in which our experience of physical phenomena must be handled if we are to achieve an adequate synthesis of it. They are most naturally expressed in experiential, not abstract, terms. They are statements about the way in which the physical world, as apprehended in human perception, is constituted. It is true that these concepts, once they were clearly recognized, were, in physics, expressed in abstract mathematical terms, e.g., the commutation relations, and used in this form to obtain the quantum mechanics. But if my suggestion that similar concepts might prove fruitful in the life sciences should be borne out, I would not expect that they would prove to be analogously expressible in mathematical terms, but rather that they would be applied in ways which correspond with the normal means of scientific investigation which already characterize the several fields involved.

A more profound question raised by Professor Fein concerns the extent to which any given concept in science is characteristic of ourselves as perceiving subjects, and the extent to which it is characteristic of the nature of phenomena apart from our mode of perception. This question, of course, goes well beyond considerations of complementarity and indeterminacy and ultimately applies to all knowledge, scientific or otherwise. The point of view from which my paper is written is a purely pragmatic one and is, as Bohr expresses it in one of the passages I quoted, "only an endeavor to clarify the conditions, in each field of knowledge, for the analysis and synthesis of experience."

WILLIAM G. POLLARD

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Negative Coefficient of Expansion of Stretched Rubber

WHILE picking up our children's toys from the livingroom the other day, I threw an old, half-deflated balloon in the fireplace to watch it expand and burst. I reasoned that the side nearest the fire would become soft, expand, and break, but, for the moment, I had forgotten that the thermal coefficient of expansion of stretched rubber is negative. To my surprise, the side nearest the fire did not become soft enough to expand but violently contracted instead. This I could observe because the balloon was of the multicolored (marblized) variety.

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An International Language for Science?

IN the September number of this *Journal*, Forrest F. Cleveland discussed under the heading "Book Reviews" the necessity of a neutral and easy international language for publication of scientific results in general, and the merits of a certain linguistic project "Interlingua" in particular.[1] I heartily agree with what F. F. Cleveland writes about the advantages of the introduction of such international language in general. Many of you who have attended international conferences in which French or some other foreign language was the official medium know how much simpler it would be to follow the scientific discussions if the language difficulty were less formidable.

I may add another important argument. In the past centuries Latin has been used as the international language of science, later French, not so long ago German, and quite recently (since about 1933) English. In fact, since Latin, because of its difficulty, lost its position as a neutral means of communication, the national language of the country producing the greatest number of important scientific publications has always served as *the* international language of science. This, however, is not always the same country, and we therefore cannot expect English to remain *the* international language of science for all centuries to come. Rather than wait till we all have to learn Japanese, Spanish, Russian, Danish, Chinese, or who knows what language next, we should anticipate this change of international language by returning to the idea of a neutral

language, avoiding the irregularities of grammer which made the now obsolete Latin inappropriate. Acceptance of a new easy and neutral language for science, however, can be achieved only gradually and only if enough scientists will join in promoting its use by publishing from time to time results in such language themselves. This is what I personally—like several others—have been doing with some of my publications, not only to set an example but also to show that it can be done.[2]

On the other hand, I heartily disagree with the proposal that we use "Interlingua," or any of the other hundreds of language projects that are not real languages. A language that is not spoken and thought by a group of people is not a living language, and nobody could readily express himself in a "paper" language during a debate in an international conference. Also, it is practically impossible to interest large numbers of scientists in spending their time learning a language for which they have no use and in which not even some literature exists. Most of these so-called international "languages" have never grown nor do they show their ability to grow beyond the scope of a project on paper. A couple of them may have been learned by a few people, but never to the extent of becoming living languages with character of their own.

Exceptions to this rule up to the present have been only two. Volapük, when it first was published, caught the fancy of millions and for a short time was learned by many. However, its grammar was far too complicated, and it broke down completely when the first international congresses of Volapükists were held and the pronunciation turned out to be too hard because of its eight different vowels.

The second exception is Esperanto. The start of Esperanto in the wake of the failure of Volapük was slow. Progress, however, was rapid from the time the first Esperanto congresses were held and fluent conversation in Esperanto was shown to be an easy matter.[3] Esperanto, contrary to Volapük, did not die a natural death but developed to become a spoken language. The author of this Letter has often spoken it himself with foreigners from many countries. During such conversations—or when writing a letter in Esperanto—one need not artificially translate into Esperanto, but one can readily think directly in this language. Esperanto, therefore, cannot be put on one line with Occidental, Ido, Interlingua, or whatever those other projects may be called, for Esperanto, contrary to these *projects*, has become a real and live *language*.

Remarks that this or that word, rule of grammar or of spelling in Esperanto might better have been chosen differently sound just as useless and a lot less convincing than similar remarks on the flaws of English, whose spelling and pronunciation are scarcely less than farcical. It is quite possible that some Interlingua people think that "lingua" looks neater (to them personally, that is!) than the Esperanto word "lingvo," but the introduction of an international language is not helped by trying to promote one's own taste and making new projects (of which Interlingua is one of the newest). It is only helped by a joint endeavor of all of us to push ahead the one existing well-established neutral international language, which, by the way, is as free from flaws as one can reasonably expect of any scientifically man-made language—that is, a lot freer from flaws than man-made so-called "natural" languages. Moreover, it has a well-developed world literature, of which I personally have over a hundred volumes on my shelves—a fraction of the total. This is but one of the reasons why knowledge of Esperanto can be worth while right now. It also offers possibilities of much wider international contacts than can easily be obtained otherwise.

If these lines can convince anybody that our personal studying of Esperanto is at least as worth while and important for the future of the world as campaigning for control of atomic energy and the like (and you'll find that it is much more fun, too), then I'll gladly answer personal requests of information on Esperanto courses, dictionaries, textbooks, literature, magazines, journals, and societies.

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[1] F. F. Cleveland, Am. J. Phys. **20**, 382 (1952).
[2] *Text in Esperanto:* Prog. Theoret. Phys. **3**, 460 (1948); **4**, 165 (1949); **6**, 202 (1951). Also, Scienca Revuo **1**, 98, 100, 106 (1949); **3**, 61, 67 (1951).
Abstract in Esperanto over English text: Physica **6**, 849, 870, 887 (1939); **7**, 305 (1940); **12**, 1, 17 (1946).
[3] There have been severe set-backs during each of the world wars and because of persecution of Esperantists by Hitler wherever he was in power, and (since about 1936) also in Russia. Recently, there has been quite a comeback of Esperanto in many countries.

Further Demonstrations in Analytical Mechanics

I PURSUE in this note my ambition to *demonstrate* certain of the classical problems of analytical mechanics, and I report here two which lend themselves elegantly to very close approximation. Indeed, these demonstrations possess a sensible beauty which can be discovered only by seeing them actually work. If the charge against them (as demonstrations) is that they are simple, I must confess that I am always intrigued by the elegant simplicity of Nature.

Problem 1: Find a point P on a vertical circle such that the time of descent along a radius to the center is the same as along a chord to the lowermost point of the circle. It is easily shown that the point P lies 60° from the top of the circle.

Demonstration: On a sheet of plywood 4 ft by 8 ft, standing on end, describe a semicircle with the mid-point of a long edge as center O. Draw a vertical diameter; call its lower end A. With a protractor locate point P. At points O, P, and A set wood screws or screw-eyes and attach piano wire along PO and PA, the wire carrying small riders. (A double electric lead connector makes a very good rider, being a metal cylinder about 2 cm long with a smooth hole along its length.) The wire can be drawn up taut by turning the screws or screw-eyes as turnbuckles. This arrangement gives a path of 4 ft along a radius and about 7 ft along a chord. On simultaneous release of both riders at P, it is observed that both arrive simultaneously at O and A. Indeed, with one's back on the experiment the simultaneity of impact is easily detected. Since the time of

descent is really too short for safe clocking, the simultaneity of descent as detected by the eye or ear is safe evidence.

Problem 2: Show that the time taken by a particle to slide down any chord which begins at the highest point of a vertical circle is constant. It is easily shown that the time is given by $2(r/g)^{\frac{1}{2}}$, where r is the radius of the circle.

Demonstration: The board used in *Problem 1* can be fitted with screw-eyes and chords bearing riders. Too many at once, however, crowds the riders at the highest point. The simultaneous arrival at the circumference is an engaging thing to witness.

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Some Remarks on D. Bernoulli's Formula

THE paper of G. A. Lindsay on "Pressure Energy and Bernoulli's Principle"[1] invites a commentary. First, we may observe that the name of Bernoulli's principle is usually attributed to the very important theorem designated otherwise as the principle of virtual displacement or work. This was first proposed intuitively by John Bernoulli (1667–1748) and afterwards elaborated with more rigor by Lagrange in his system of mechanics. But what G. A. Lindsay's paper is concerned with is the remarkable formula demonstrated by Daniel Bernoulli (1700–1782),[2] the paramount importance of which has been realized only in our time. This formula can be obtained from two starting points which, for most physicists, may seem very far apart: (1) the principle of conservation of energy, and (2) Newton's second law.[3] It would be, of course, faulty to consider the pressure p as an energy. It is but a factor, i.e., the intensity, the volume being the extensity, of a form of energy called volume or spatial energy. But the demonstration of D. Bernoulli's formula from the principle of conservation of energy can be done quite correctly and has so many merits that it should be included into syllabuses. The point that it can be also deduced starting from Newton's second law is very interesting and can be most easily explained on the lines of my system of energetics.

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[1] Am. J. Phys. **20**, 86 (1952).
[2] At least three members of the Bernoulli family gave distinguished services to physics and mathematics.
[3] See, for example, E. Schmidt, [*Thermodynamics*, (Oxford University Press, London 1949), English translation, pp. 324, 332], who was rather startled by this fact.

On an Aspect of Demonstration Experiments

THE place which demonstration experiments occupy in physics instruction needs no discussion. As instructional devices they are probably unexcelled. In the hands of a good demonstrator, accompanied by the appropriate "language," even the most elementary experiment can be stirring. There are, of course, as many ways to demonstrate an experiment and as many ways to "say the physics" as there are teachers, but one very important aspect exists of which, I have repeatedly observed, teachers fail to take advantage. This aspect I might title "Questions to be Asked." During the demonstration the teacher usually recites what he has, what he is going to do, and what will be observed to happen. And it usually happens. (If it does not, some good physics can still be taught.) Now the detail to which I refer explicitly is this: it is highly instructional to ask here and there, "Will *this* happen?", or, "Why will *this not* happen?" I can best illustrate the point by reference to an elementary demonstration.

Consider a demonstration of linear momentum. I use a track made of angle-iron on which reside a string of billiard balls. One ball is taken up the incline and released. After impact, one ball runs off from those at rest. Two balls are taken up the incline; after impact, two balls run off from those at rest. Suppose now that more are taken up the incline than remain on the level track. What will happen? How many will go away? It is surprising to discover how puzzling this question is. But let's go on. After it is shown that as many go away as came down, I like to ask this: with two coming down, say, we found two going away after collision. The mv is taken care of. Now could not the mv be accounted for by *one* ball going away with a velocity $2v$? And with this I repeat the experiment several times, but this does not happen! Why? (I once had a good student who had read about entropy. Regarding the billiard balls, he was of the opinion that it might eventually happen, just as the water in a glass might suddenly freeze on a hot day, or the original order in a shuffled deck of cards be gotten again.)

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Magnitude of the Newton

OUR teaching methods should utilize as many of the five senses as is practicable. Whenever we make use of the sense of touch, we add to the ability of the student to grasp the idea under consideration. To present the student with an appreciation of its magnitude, we have constructed a demonstration "newton."

The newton is defined as the force required to give 1 kilogram an acceleration of 1 meter/sec^2. The acceleration is approximately one-tenth that of gravity, so the earth pulls 0.1 kilogram with a force of approximately 1 newton. The error is 2 percent. The precise mass required for an earth-pull of one newton is $1/g$ where g is the *local* value of the acceleration of gravity. This mass is more nearly 102 grams.

To make our demonstration "newton," we added the appropriate number of grams of solder to a brass 100-gram weight so as to obliterate the original 100 grams. Next we stamped on the solder the phrase "1 newton at U. of Ala." When we pass around our "newton" in elementary classes, it becomes a "real" physical quantity to the student, not just another theoretical quantity to complicate life.

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ANNOUNCEMENTS AND NEWS

Theory of Superconductivity. M. VON LAUE, translated by L. MEYER AND W. BAND. Pp. 140+x, Figs. 36, 15½×23½ cm. Academic Press, New York, 1952. Price $4.00.

The theory of which M. von Laue, Nobel laureate in physics, has so carefully written is the phenomenological electrodynamics which results when the equations of Maxwell are applied to the empirical equation for a superconductor by Fritz and Heinz London. The author has collected a number of interesting problems on variously shaped superconductors about which the field distribution, current distribution, and related effects are worked out. The experimental physicst will find this 140-page book a valuable addition to his reference collection. The translation was completed in 1949, but there are references and remarks about work published in 1950. The theory and mathematical steps are well written and easy to follow. Professor von Laue considered the quantum-theoretical efforts made on the detailed picture of superconductivity as premature and has confined his writing to only the classical theory. The ponderous tensor notation which the author presents to the reader in the early pages is avoided in most of the application to specific problems.

The praise which this book received from Fritz Sauter [see *Naturwissenschaften* **37**, 263 (1950)] that the phenomenological theory is "in ersten Linie mit dem Namen von Laues verbunden" is not wholly justified. In part this impression might be given because von Laue does not burden himself with giving prior credit. Thus, for example, the experiments on superconductors at very high frequency had been carried out by H. London, who also first gave the proper theory for the energy absorption. But in the book the inaccurate studies by McLennan and co-workers are evaluated with a theory claimed by von Laue. The last chapter of the monograph, "A Nonlinear Extension of the Theory," deals with some highly controversial theory which was stimulated by Heisenberg's theory of superconductivity.

Chapter 19 deals with the phase transition into the superconducting state and discusses the intermediate state. No use is made of the concept of a surface energy between the superconducting and normal state. The interested reader will welcome a supplement to this chapter by reading von Laue's article in *Annalen der Physik* **10**, 296–316 (1952). The editors of the journal were cooperative in requesting Professor F. London to add some helpful comments in response to von Laue's paper and in this way agreement on the surface energy has been reached. The unconvincing work by Stark, Steiner, and Schoeneck received some attention and yet such things as the classical work by Kikoin and Guber on the gyromagnetic ratio of a superconductor is not included in the monograph.

C. F. SQUIRE
Rice Institute

Proceedings of the London Conference on Optical Instruments 1950. Pp. 264+xv, Figs. 103. John Wiley and Sons, Inc., New York, 1952. Price $7.00.

The present volume is a compendium of papers resulting from a topical symposium on modern optical instruments held under the auspices of the International Optical Commission in London during July, 1950. This conference on optical instruments was designed to follow in the tradition of the great International Optical Conventions of 1905, 1912, and 1926. The objective of the conference was principally the design and application of modern optical instruments such as photographic objectives, reflecting microscopes, grating spectrometers, phase contrast microscopes, spectrophotometers, reflecting telescopes, and other modern developments.

The introductory remarks by Sir T. Merton aptly point out that optics today is in a period of extended maximum activity and is actually becoming chiefly an applied science. The tremendous accumulation of instrumental research during the past decade makes the need for a volume such as this one imperative in order to bring under one cover at a reasonable cost a condensation and integration of those developments which are often published in such obscure literature as patent descriptions or are otherwise scattered throughout the scientific literature of the world.

Three experts contribute to the chapters on "Photographic and Projection Lenses." R. Kingslake of the United States writes comprehensively on "Some Recent Developments in Photographic Objectives." Excellent descriptions are accompanied by numerous illustrations of various lens types. Conventional designs of double, triplet, four-element, Petzval-type lenses as well as wide-angle lenses, telephoto, and zoom lenses are discussed. Reference is made to the recent application to the photoraphy of catadioptic systems involving a reflection element and a correction plate. H. H. Hopkins of Great Britain relates in detail the theory and properties of the zoom lenses in the section entitled "A Class of Symmetrical Systems of Variable Power." In the chapter on "A New High Aperture Photographic Objective Having a Spherical Field," A. Warmisham (Great Britain) points out the great flexibility and precision attainable with the Schmidt-optics type of lens.

A truly recent development in optics is the reflecting microscope. A purely reflecting system is completely achromatic, and only the reflectivity of the surfaces limits the useful wavelength range. A Burch microscope has been used from 240 mμ in the uv to 14μ in the infrared. These three chapters are ably contributed by R. Barer (Oxford), A. Bouwers (Netherlands), and D. S. Grey (United States), and emphasize so well how the biologist and the physicist can profitably cooperate in the field of biophysics.

It is always a thrill in science when one witnesses the invention of an instrument which will improve the order

of measurement by a full magnitude. Such is the case for the echelle grating discussed by G. R. Harrison and C. L. Bausch of the U. S. A. in the first chapter under "Gratings and Grating Instruments." "A Grating Spectrograph with Interchangeable Receivers for Industrial Purposes" is discussed by E. Ingelstam (Sweden), and "Some New Mountings of Diffraction Gratings and the Plane Mirror" by E. Hulthen and E. Lind of Stockholm.

F. W. Cuckow of Great Britain gives an excellent description of the advantages of the phase-contrast microscopic method applied to the study of surface structures in metallurgy. The advantages are clearly shown by comparing an ordinary photomicrograph with an electron micrograph and with a phase-contrast photomicrograph of alloy steel, each at the same magnification. The description of the apparatus and of the techniques necessary is very well illustrated. M. Francon of France presents the recent progress in phase-contrast microscopy of the Institute of Optics in Paris. He describes a mirror reflection scheme which is absolutely stigmatic and aplanatic to be placed between the sample and the objective of the microscope to give an image in phase contrast without the use of a phase plate.

A comprehensive review of spectrophotometry in the United States is presented by S. S. Ballard which would serve as an excellent guide to anyone attempting to reach a decision on the type of instrument best suited for a particular application. The photometric equipment of the Institute of Optics in Paris is thoroughly described by F. Desvignes. An excellent discussion of the slit-width dependence of the output is described in French and presented graphically.

A detailed discussion of the modern trends in reflecting telescope design is presented by E. H. Linfoot of Great Britain and recent Swiss developments by W. Lotmar. Nearly every modern type of instrument will be found described in these two chapters including the Baker-Schmidt type for the ADH-observing station at Blomfontein and a similar instrument for the St. Andrews Observatory.

Erik Bergstrand of the Swedish Geographical Survey reports in Chapter 17 on an apparatus intended for measurements of distance by high frequency light signaling. It is based on the principle used by Fizeau for terrestrial determinations of the velocity of light modified by making use of a Kerr cell and a photocell. Conversely, if the distance is known, the velocity of light may be determined. The method and results are thoroughly described and illustrated.

Chapter 18 is concerned with apparatus for the photometry of optical instruments by J. Guild of Great Britain. He describes apparatus designed at the National Physical Laboratory for measuring such photometric properties of telescopes and photographic lenses as the paraxial transmission factor, the relative brightness of various zones of the field, and scattering and multiple reflection effects.

The concluding chapters are written by S. S. Ballard of the United States on "Synthetic Optical Crystals," by I. C. Gardner of the National Bureau of Standards on "New Types of Optical Glasses Available in the United States," and by H. C. Raine of Great Britain on "Plastic Glasses." These three chapters rather thoroughly summarize the developments during the last decade on synthetic optical materials. It will be noticed that Ballard's article does not contain a bibliography, but he points out that the firm of Baird Associates, Inc., will shortly publish a handbook of the properties of synthetic crystals in the *Journal of the Optical Society of America*. At this writing (July, 1952) this handbook has not appeared in that journal.

The book is exceedingly well written throughout and profusely illustrated. Each chapter contains an extensive bibliography which will be a great time-saver for the researcher. An unusual feature in the book is the presence of a blank page for notes at the end of a couple of the chapters. This is an excellent reference book for the teacher of optics whether teaching at the elementary or at the advanced level. The book reveals the tremendously rapid progress being made in optical instrument design today and should be inspiring reading to the student of optics as well as a challenge for him to contribute to these exciting events. Perhaps another International Optical Conference may be held in a few years to record these events in a treatise as well done as this one.

DAVID M. GATES
University of Denver

Low Temperature Physics. F. E. SIMON, N. KURTI, J. F. ALLEN, AND K. MENDELSSOHN. Pp. 132, Figs. 59. Academic Press, Inc., New York, 1952. Price $3.50.

Ten years ago there were fewer than a dozen laboratories in the world equipped for research in low temperature physics—today there are more than fifty in the United States alone. Such institutes represent fairly formidable investments in time and money; why then has such an interest, almost unprecedented, arisen in so short a time? This small book, aimed principally at the nonspecialist, provides much of the answer.

The four chapters, of roughly equal length, represent a series of lectures, one by each author, delivered before the Royal Institution in February and March, 1950. The writers are distinguished British physicists, each of whom has made substantial contributions to cryogenic physics, and no more authoritative group in this field could easily be assembled.

The first chapter, by Simon, is a rather general survey (a sort of introduction to what follows) covering, in simple nonmathematical language, a diversity of low temperature phenomena. He brings out very clearly the predominant role of "macroscopic quantum effects" in the phenomena occurring at liquid helium temperatures. A number of enticing suggestions for further experimentation in little-exploited branches of the subject are included.

Chapter II, by Kurti, deals with ultra low temperatures—the technique of adiabatic demagnetization. This is a very readable account, again with a minimum of mathematics, of the principles and practices involved in the production and use of temperatures within 0.01 degree,

and less, of absolute zero. A particularly clear exposition is provided of the proposals involving production of temperatures around 10^{-6} °K by two-stage demagnetization, using electronic and nuclear spins, a technique which is currently under investigation in several laboratories.

The third chapter, by Allen, discusses the strange phenomena associated with the "quantum-liquid" helium II. This, again, is a very clear, commendable, and critical account of most of the discoveries up to 1950. The author suffers the fate of all reviewers in a fast-moving subject—a lot of the exciting work on He^3 was published after the lectures had been given.

The book closes with Mendelssohn's chapter on superconductivity; again an excellent account considering the size of the subject and the space available. Mendelssohn closes the chapter with a rather moving appeal for a "new concept in physics" to explain the so diverse and yet so similar properties of superfluidity and superconductivity.

Despite the multiplicity of authors, the book has a surprisingly coherent style. And the formal text is illuminated (in the English manner) with many simple and intriguing observations. Thus, "at 0.03°K the vapor pressure would be so small that in a volume as big as our whole galaxy not a single atom would be in equilibrium with liquid helium" (Simon), or "the heat capacity of 10 gm of iron ammonium alum at 0.05°K is about the same as that of a quarter-ton of copper . . . at the same temperature" (Kurti).

C. T. LANE
Yale University

General Education in Science. Edited by I. BERNARD COHEN AND FLETCHER G. WATSON. Pp. 217+xviii, 14×21 cm. Harvard University Press, Cambridge, 1952. Price $4.00.

Anyone who has followed the growth of general education in the sciences can hardly fail to be stimulated by a reading of this small volume. Fifteen scientists, presenting their views before a Workshop in Science in General Education, can hardly be expected to concur on many aspects of the subject, even if nine of them are from Harvard! They are unanimous, however, in believing that science has played and will play an important part in our civilization and that traditional introductory courses in the sciences fail to convey an adequate understanding of the nature and role of science to those students who will major in some nonscience field. This indictment of traditional science courses is laid by some to the increased specialization in the sciences during the past half-century and the consequent narrowing of the training received by college teachers. René J. Dubos of the Rockefeller Institute for Medical Research decries the widespread equation of science with technology by most people, and their total ignorance that science can enrich life with new esthetic, emotional, and intellectual satisfactions. He traces this attitude to the failure of science in the nineteenth century, through positivist philosophy, to solve the riddle of the universe and existence.

Several of the contributors give their ideas on what are the most important aspects of science that should be presented to nonscience students, and the most effective methods. They are generous enough to acknowledge that no single approach has all the answers and that is it more important for a teacher to use one in which he feels a natural interest and some ability to prepare himself. The case method, the problem approach, critical thinking in science, the emphasis on history, or on philosophy, are all treated by teachers who have developed and used them. Anyone contemplating the organization of a general education science course should place this volume on his personal list of required reading. Worth-while thoughts about what such teachers should know about philosophy are presented. Three of the contributors make valuable suggestions on how a teacher may enlarge his background in historical materials, either for developing historical cases or merely to convey a sense of history. Even the worrisome matter of evaluation is treated helpfully.

To strive for an understanding of what science is and what it does for a sense of its history or a glimpse into some of its philosophical implications—apart from mere knowledge of some of its facts, principles, and applications—is what is being attempted in these various courses for nonscience students. But this reviewer cannot help wondering about the remaining 31 percent (at Harvard) of "science" students. How do *they* acquire this liberalizing "understanding of science" which these contributors feel is so important. Its absence from the standard introductory courses must be presumed by the very development of general education courses. Perhaps it is acquired at advanced undergraduate or graduate levels. (The narrowness of modern graduate work in science raises some doubts on this point.) But very few of the 31 percent of freshman carry through to graduate study in science. A number follow engineering, others transfer to a nonscience major before the bachelor's degree. In short, a large block of students is still not receiving some of the benefits of general education in science.

This is a difficult situation to deal with. Experience has shown that occasional footnotes or side remarks are of little value. A real impression must be made. But training in current science and technology is a time-consuming task, and, where a gap can be pried open, most people feel that a little more of the humanities or the social sciences should be wedged in. Two of the contributors to this volume have made brief suggestions to meet this problem. Paul B. Sears, a biologist at Yale, claims that the objectives of both general education and basic preparation in the sciences could be met by a unified and reorganized two-year sequence for science and nonscience students alike. Sidney J. French, a chemist at Colgate, has interesting ideas for reorgainzation of science courses at the upper levels. Both admit, along with this reviewer, that there would have to be sacrifice in a number of places. Judged by the current product, our science training programs are good. Is it too much to expect that, with careful planning, they could be improved by including some of the objectives of general education in science?

CLEMENT L. HENSHAW
Colgate University

Impact of Science on Society. United Nations Educational, Scientific and Cultural Organization, Paris, France. Quarterly journal. Annual subscription $2.50, single copies 75¢ (in U.S.A.). Distributed in various countries by UNESCO sales agencies (see below).

One of the most frustrating features of the life of today's scientist or science teacher is the universally acknowledged difficulty of even scanning the material offered by the many journals in his own and related fields. It would therefore appear to require some courage to bring to the attention of the readers of the *American Journal of Physics* still another publication that may legitimately claim their notice. However, even apart from its inherent excellence and the unique function it was created to serve, there need be no hesitancy in recommending to scientists in all fields—and indeed to every thoughtful person who cares to understand the importance of science in the affairs of men—this new and valuable quarterly produced by the Department of Natural Sciences of UNESCO. For it is precisely the inclusion of such a journal in the busy scientist's regimen of reading that will balance the many technical items he must peruse and that is capable of reminding him of the place and importance of his special field in the life of our time.

As the name and sponsorship of the publication suggest, the purpose of *IMPACT* is to focus attention throughout the world on science as a powerful social force. Much excellent work is being done by social scientists in studying and appraising the consequences for society of past accomplishments of science and technology. The editors of *IMPACT* feel, however, that the delay in the social study until after the consequent processes are already in motion is unfortunate. What they hope to accomplish through this publication is to stimulate social scientists, educators, economists, legislators, and leaders of thought to consider the possible *future* effects of *present* scientific discoveries. It must be granted that in many cases these consequences can be foreseen, at least in a general way, at the time when new scientific or technical advances are announced.

These aims are realized through the medium of contributed articles on subjects concerned in some way with the interrelations of science and society; and a number of rather complete and detailed summaries of current books or articles on this field round out the text of each issue.

Although *IMPACT* was founded two years ago, it now appears with completely revised content, style, and format. The Summer, 1952, issue which is at hand for review includes the following articles: "The Education of the Public in Science," by I. Bernard Cohen; "The Bulletin of the Atomic Scientists—A Magazine for Science and Public Affairs," by Eugene Rabinowitch; and "A Basis for Social Physics," by John Q. Stewart. Professor Cohen's paper recognizes the disturbing problem arising from the fact that modern means of communication give the average person much information about science but contribute little to its understanding; and he discusses this question in the light of his own experience in teaching science to college students. This article is one that every teaching physicist will wish to read with care.

Because the *Bulletin of the Atomic Scientists* and the organization which it represents have brought to the attention of American scientists their social responsibilities in a way never before attempted, the editors invited Dr. Rabinowitch to prepare his description of the *Bulletin* and its aims. This article will be read with great interest particularly by foreign scientists who are not yet acquainted with the Bulletin or who are unable to obtain it regularly.

Readers of this *Journal* probably are already familiar with the name of the subject discussed in the article of Professor Stewart through his paper in the May, 1950 issue. Sharing the conviction of many scientists, both social and physical, that the social sciences will eventually be developed toward the status of exact disciplines involving quantitative laws and generalizations, Dr. Stewart has himself succeeded in discovering several of these relationships. In the paper in *IMPACT* he sketches a program for working out a "physics of society" and examines some of the highly suggestive comparisons that can be drawn between certain physical and social phenomena.

Subsequent issues of the magazine are to contain article on such subjects as "National Development Based on Science: The Case of Denmark," "The Parliamentary and Scientific Committee in Britain," "Population Movements among Animals and People," "The Origins of the Popularization of Science in France," and "Cybernetics and Civilization," written by authorities from various countries. To judge from the contents of the current issue and the offerings promised in future ones, not the least of the uses of *IMPACT* to the teacher of science will be to furnish material for meaningful classroom discussion, particularly in courses for nonscience students.

IMPACT is published in French as well as in an English edition. In the United States, the English edition is available through the Columbia University Press, 2960 Broadway, New York City.

IRA M. FREEMAN
Rutgers University

The Theory of Relativity. C. MØLLER. Pp. 386+xii. Oxford University Press, London, 1952. Price $7.00.

The publication of a book on the theory of relativity is a rare event these days. The appearance of the present book is to be welcomed, however, not only for this reason, but also because it provides a serious, carefully written, and unusually thorough account of the theory that can be highly recommended for the student of theoretical physics. For that matter, it can be recommended for anyone with an adequate knowledge of mathematics and physics who would like to become acquainted with the beautiful structure of the relativity theory and its applications.

Well over half of the book is devoted to the special theory of relativity. The remainder deals with the general theory and includes a brief introduction to relativistic cosmology. Although the book concerns itself only with the classical relativity theory, it is evident that its author is interested in the quantum theory and the application of relativity theory to the latter. Thus, the motion of waves

receives considerable discussion, including a comparison of the phase velocity of a wave with the velocity of a particle. This paves the way for the introduction of the ideas of de Broglie concerning particles and waves which played such an important role in the development of wave mechanics. There is a derivation of the expression for the Thomas precession, the precession of a vector associated with a particle, such as the spin of an electron, when the particle undergoes an acceleration not parallel to its velocity. As an example of a field, the scalar meson field is briefly treated.

As he points out in the preface, the author tries to make clear the physical meaning of the quantities occurring in the relativity theory by giving a three-dimensional formulation, in addition to the usual four-dimensional one. For example, the fundamental or metric tensor which determines the geometry (and gravitation) of the four-dimensional space-time continuum is split up into a three-dimensional metric tensor, a gravitational "scalar potential," and a gravitational "vector potential."

The author has performed a valuable service in bringing together many of the important results obtained during the early part of this century by Einstein, Poincaré, Minkowski, and others. Physics work tends to get buried under the work of subsequent generations. How many of the younger physicists are familiar with the disagreement between Abraham and Minkowski concerning the proper expression for electromagnetic momentum density in a material medium? Møller presents arguments supporting Minkowski's standpoint. These might well be noted by future writers of books on electromagnetic theory.

In the general theory of relativity both geometry and gravitation are described by the expression for the interval or line element between two neighboring events in terms of the coordinates and their differentials. In principle, this interval can be measured by a suitable procedure involving the use of clocks and measuring rods. In this connection, on p. 232, the author writes: " . . . if all measuring-rods in the neighborhood of a given point for one or another reason were dilated at the same rate independently of the material of which they are made, it would be impossible to observe this dilation and in an unambiguous way to correct for it. Therefore, there is no well-defined meaning in the statement that such an expansion of the standard rods has taken place, and from a physicist's point of view the metric tensor and the geometry obtained by measurements with the natural standard rods must be the 'real' geometry on the surface." One wonders about this. If we suppose that, by assuming a suitable expansion of the measuring rods at each point, one could simplify the geometry, why not assume such an expansion? This, however, would lead us to developments lying outside the scope of the book.

NATHAN ROSEN
University of North Carolina

Vector Analysis. EARL C. REX. Pp. 88+viii, Figs. 45, 8½×11 in. Lithographed. Wm. C. Brown Company, Dubuque, 1952. Price $3.25.

This text covers vector algebra, dyadics, and vector calculus. Fifty-three illustrative problems are worked out, and there are numerous exercises, many of a routine computational nature. No answers are given. At the end of each chapter is a lengthy summary, which, in some cases, includes additional discussion and new proofs.

The third chapter contains two innovations, both apparently original with the author. The first is the use of vector equations to describe all of the points interior to and on a surface. Applications of this technique are given in both the third and fifth chapters. The other innovation is the definition of "dot" and "cross" products of complex numbers.

The author appears to have put a great deal of thought into many parts of the book, particularly on vector algebra. In spite of this, the book has flaws, of which the most serious are numerous errors and frequent lack of rigor. In his derivation of the scalar and vector products of two vectors in component form from the definitions, the author includes every step, except that he fails to mention (let alone prove) that these products are distributive with respect to addition, and that his derivation depends on this fact. Most of the proofs are given in great, not to say excessive, detail. A notable exception is the divergence theorem, which the author considers "almost axiomatic." This might, perhaps, be justifiable, had the integral definition of divergence been used. Divergence is, however, defined using the del operator in differential form, and the discussion of the physical meaning of divergence is obscure and insufficient to render the divergence theorem obvious.

The following are typical of the errors which occur. On page 19, the vector equation of a hyperbola, in the standard position, is given as $r=ai\cos\theta-bj\sin\theta$. In illustrative example 43, the direction of the magnetic field of a long straight wire carrying a current is taken to be radially outward from the wire. In illustrative example 51, the author obtains the value zero for $\int_{-\infty}^{\infty}dr/(1+r)^2$ because he overlooks the singularity of the integrand at $r=-1$. It is unfortunate that naiveté of this sort permeates the entire work.

The author uses unusual terminology in many places, and frequently introduces terms without definition. The most glaring instance of the latter is the use in Chapters I and II of the word "component," which is not defined until the end of Chapter III. The discussion is frequently obscure, and the wording ranges, on occasion, from clumsy to ungrammatical. There are a number of misprints.

MARY H. PAYNE
Michigan State College

RECENT MEETINGS

American Physical Society
New England Seciton

The fall meeting of the New England Section of the American Physical Society was held at the University of New Hampshire on October 25, 1952. The following program was presented:

Welcoming remarks. President ROBERT F. CHANDLER, JR., *University of New Hampshire.*

Symposium on High Energy Physics

Design considerations for very high energy accelerators. M. S. LIVINGSTON, *Brookhaven National Laboratory* and *Massachusetts Institute of Technology.*

New particles in cosmic rays. R. W. WILLIAMS, *Massachusetts Institute of Technology.*

Interaction of protons with free and bound neutrons in the 100-Mev region. KARL STRAUCH, *Harvard University.*

Address of Retiring Chairman

Some optical filters. NORA MOHLER, *Smith College.*

Contributed Papers

1. An inexpensive, easily controlled process for developing 400-micron nuclear emulsions. MARY B. SUMMERFIELD, *University of California, Berkeley.*

2. Quantum-mechanical methods in classical physics. DAVID L. FALKOFF, *Brandeis University and Massachusetts Institute of Technology.*

3. Activation energies of the selenium-tellurium alloys. HARRY H. HALL, DANIEL T. HEDDEN, and THOMAS J. TURNER, *University of New Hampshire.*

4. Crystal diodes in the electronics laboratory. IRVING L. KOFSKY, *Smith College.*—Some experiments using germanium diodes, suitable for use in the undergraduate electronics laboratory, were described. Among these are the familiar tubeless unpowered crystal radio; a simple clipper circuit for producing trapezoidal waves, which are used in an experiment designed to illustrate the response of several circuit elements to pulses; and simple voltage- and frequency-multiplier circuits.

The question of where in the traditional electronics course to introduce the newly developed semiconductor substitutes for vacuum tubes was discussed.

W. M. PRESTON, *Secretary*

The Tape Recordings of Important Speeches

BY

E. U. CONDON, K. K. DARROW, E. FERMI, J. C. SLATER

Free tape recordings are available on loan to members of the Association who wish to use them for presentation to classes, seminars, or science clubs. The subjects are four of the six lectures presented as a Symposium on Physics Today at the Twentieth Anniversary meeting of the American Institute of Physics, Chicago, Illinois, on October 25, 1951, as follows:

The atom. E. U. CONDON.
Physics as science and art. K. K. DARROW.
The nucleus. E. FERMI.
The solid state. J. C. SLATER.

Professor Slater's speech has been added to the list since our first announcement.

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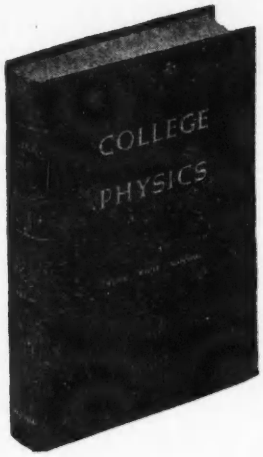
Prospective borrowers may consult the published papers in *Physics Today*, November, 1951, January, 1952, and March, 1952.



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